

TRYING TO UNDERSTAND MASS

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Abstract

We try to understand how particles acquire mass in general, and in particular, how they acquire mass in the standard model and beyond.

1 Introduction

At the end of my talk at the Galapagos World Summit on Physics Beyond the Standard Model¹ I mentioned, in passing, three crazy ideas.² In this talk I elaborate on one of them.

The origin of mass is a mystery. Particles may acquire mass, or change their mass, when they interact with a medium. Examples are photons interacting with free electrons in a plasma or in a metal, photons interacting with Cooper pairs in a superconductor, electrons propagating in the periodic potential of a crystal, and photons propagating in a waveguide.

The symmetries of the standard model Lagrangian prevent adding “by hand” mass terms for the fermions or bosons. These particles acquire mass due to their interaction with the Higgs field.

The limit $m \rightarrow 0$ is singular in the sense that phenomena present with $m \neq 0$ is absent with $m = 0$. Examples are the longitudinal polarization of W^+ , W^- and Z , the coupling between the left and right Weyl components of the Dirac field, and the Cabibbo-Kobayashi-Maskawa matrix.

The origin of mass is the next frontier in high energy physics. In this talk I try to understand how particles acquire mass in general, and in particular, how they acquire mass in the standard model and beyond.

¹San Cristobal Island, 22 – 25 June 2006.

²Ideas developed incompletely, so that their usefulness (or correctness) has not been established.

2 Dispersion relation

Let us consider the wave

$$\propto e^{i(\vec{k}\cdot\vec{r}-\omega t)} \quad (1)$$

with the dispersion relation

$$\omega_0^2 = \omega^2 - k^2 c^2. \quad (2)$$

ω_0 and c are constants. Let us give some examples. For an electromagnetic wave in vacuum, $\omega_0 = 0$ and c is the velocity of light. For a sound wave, $\omega_0 = 0$ and c is a velocity of sound. For an electromagnetic wave in a plasma or in a metal, $\omega_0 = [e^2 n / (\epsilon_0 m)]^{1/2}$ is the plasma frequency and c is the velocity of light. For an electromagnetic wave in a waveguide, ω_0 is the cut-off frequency of the mode of propagation and c is the velocity of light. For an electromagnetic wave in a dielectric, $\omega_0 = 0$ and c is the velocity of light in the dielectric. For a Dirac (spin- $\frac{1}{2}$) particle, $\omega_0 = mc^2/\hbar$ and c is the velocity of light.

To every wave (1) there is associated a particle of energy

$$E = \hbar\omega \quad (3)$$

(Planck relation), momentum

$$\vec{p} = \hbar\vec{k} \quad (4)$$

(De Broglie relation), and velocity equal to the group velocity of the wave:

$$\vec{v} = \nabla_{\vec{k}}(\omega) = \frac{\vec{k}c^2}{\omega} = \frac{\vec{p}c^2}{E}. \quad (5)$$

From the preceding equations we obtain the dispersion relation of the particle,

$$m^2 c^4 = E^2 - p^2 c^2, \quad (6)$$

where the “cut-off mass” is given by

$$mc^2 = \hbar\omega_0. \quad (7)$$

If the particle has zero mass,

$$E = pc, \quad \omega = kc. \quad (8)$$

If the particle is massive, we obtain

$$E = \frac{mc^2}{\sqrt{1 - (v/c)^2}}, \quad (9)$$

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-(v/c)^2}}. \quad (10)$$

For a particle in vacuum, c is the velocity of light, and the energy-momentum (E, \vec{p}) , given by (9) and (10), transforms the same as the space-time coordinates (t, \vec{r}) . Therefore, the energy-momentum of a particle in vacuum is a 4-vector with respect to the Lorentz group.

3 Inertia

Is the “cut-off mass” related to the “inertial mass” in Newton’s equation? Yes! Let us give an example. Consider a horn, i.e. a waveguide with transverse dimensions (x, y) increasing with z . The group velocity v_z increases with z , the corresponding particle is accelerated, and the particle momentum increases as

$$F \equiv \frac{dp_z}{dt} = \frac{d}{dt} \left(\frac{E v_z}{c^2} \right) = \frac{E}{c^2} \frac{dv_z}{dt} \approx m \frac{dv_z}{dt} \quad (11)$$

if $v \ll c$. Therefore, the “inertial mass” is equal to the “cut-off mass” m . F is the force. The horn momentum increases by the same amount per unit time in the opposite direction. $E = \hbar\omega$ is constant because the horn does not change with time. p_z is not constant because the horn varies with z .

In general, consider any system with energy E_0 in its rest frame, i.e. the frame with $\vec{p} = 0$. After a Lorentz transformation, the system has velocity \vec{V} and momentum $\vec{p} = E_0 \vec{V} / c^2$ if $V \ll c$. So the system has inertia with mass $m = E_0 / c^2$. Inertia, and the conservation of energy-momentum, are two aspects of the same phenomenon.

4 Conservation of energy-momentum

Let us try to understand the conservation of energy-momentum. As an example, we consider this experiment: we illuminate an atom with two beams of light, and observe the light scattered in the direction \vec{k}_3 with a receiver far away. The atom is at \vec{r}_j , and the receiver is at \vec{r} . The electric field of the incident beams at the atom is

$$A_1 e^{i(\vec{k}_1 \cdot \vec{r}_j - \omega_1 t)} + A_2 e^{i(\vec{k}_2 \cdot \vec{r}_j - \omega_2 t)}. \quad (12)$$

This field induces dipole, quadrupole, etc, moments in the atom, so the atom emits an electromagnetic wave. The amplitude of this scattered wave in the direction \vec{k}_3 is some non-linear function of (12). Expanding this non-linear function in a Taylor series, we find that the amplitude of the scattered wave has the form

$$\sum_{n,m} a_{nm} e^{in(\vec{k}_1 \cdot \vec{r}_j - \omega_1 t)} e^{im(\vec{k}_2 \cdot \vec{r}_j - \omega_2 t)}, \quad (13)$$

with $n, m = 0, \pm 1, \pm 2, \dots$. The electric field at the receiver is proportional to

$$\propto \sum_{n,m} a_{nm} e^{in(\vec{k}_1 \cdot \vec{r}_j - \omega_1 t)} e^{im(\vec{k}_2 \cdot \vec{r}_j - \omega_2 t)} e^{i\vec{k}_3 \cdot (\vec{r} - \vec{r}_j)}. \quad (14)$$

Note that the amplitude at the receiver has components of frequency

$$\omega_3 = n\omega_1 + m\omega_2. \quad (15)$$

Multiplying by \hbar (to get the conventional unit for energy) we obtain the equation of conservation of energy:

$$\hbar\omega_3 = n\hbar\omega_1 + m\hbar\omega_2. \quad (16)$$

In our example with positive n and m , n incoming photons of energy $E_1 = \hbar\omega_1$ scatter on m incoming photons of energy $E_2 = \hbar\omega_2$, producing an outgoing photon of energy $E_3 = \hbar\omega_3$. Note that *classical* waves interact in packets of frequency or energy.

It is convenient to include a factor $e^{i(\vec{k}_3 \cdot \vec{r} - \omega_3 t)}$ in the proportionality constant of (14). Then equation (14) becomes

$$\propto \sum_{n,m} a_{nm} e^{in(\vec{k}_1 \cdot \vec{r}_j - \omega_1 t)} e^{im(\vec{k}_2 \cdot \vec{r}_j - \omega_2 t)} e^{-i(\vec{k}_3 \cdot \vec{r}_j - \omega_3 t)}. \quad (17)$$

The part exhibited explicitly in (17) is independent of (t, \vec{r}) .

Let us now consider a crystal. The amplitude at the receiver is a sum of terms (17) over all atoms j in the crystal:

$$\propto \sum_{j,n,m} a_{nm} e^{in(\vec{k}_1 \cdot \vec{r}_j - \omega_1 t)} e^{im(\vec{k}_2 \cdot \vec{r}_j - \omega_2 t)} e^{-i(\vec{k}_3 \cdot \vec{r}_j - \omega_3 t)}. \quad (18)$$

Since, for a 3-dimensional crystal lattice, $\vec{r}_j = \vec{r}_0 + N\vec{a} + M\vec{b} + L\vec{c}$ with N, M and L integers, it follows that the condition that the waves scattered at each atom j add up in phase at the receiver is

$$n\vec{k}_1 + m\vec{k}_2 = \vec{k}_3 + \vec{G}, \quad (19)$$

where $\vec{G} \cdot \vec{a}$, $\vec{G} \cdot \vec{b}$ and $\vec{G} \cdot \vec{c}$ are multiples of 2π . Equation (19) is the Bragg law, or, multiplying by \hbar , the law of conservation of “crystal momentum”. \vec{G} is a vector of the “reciprocal lattice”. $\hbar\vec{G}$ is the momentum acquired by the crystal as a whole. In the limit of a continuous medium, i.e. in the limit $\vec{a}, \vec{b}, \vec{c} \rightarrow 0$, the only finite \vec{G} is $\vec{G} = 0$, and we obtain the law of conservation of momentum.

In conclusion, the conservation of frequency-wave vector (or equivalently, energy-momentum) is the condition for the waves scattered at different space-time points to add up in phase at the receiver, and is valid if space-time is homogeneous, i.e. if the vertex factor in the Lagrangian does not depend explicitly on space and time. Note that, for scattering to take place, the Lagrangian must be non-linear, i.e., the vertex terms must contain 3 or more fields. Since the powers in the Taylor expansion, n, m, \dots , are integers, the interaction occurs in packets of energy-momentum $(E_i, \vec{p}_i) = (\hbar\omega_i, \hbar\vec{k}_i)$ we call “particles”. Note that the concept of *particle*, and the Planck and De Broglie relations, have emerged from the interaction of *classical* waves! Quantum mechanics tells us that nature *chooses* among the various scatterings n, m with probabilities proportional to $|a_{nm}|^2$.

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Note that the Feynman rules to obtain the probability amplitude for a particular scattering is obtained by multiplying a factor $\exp[i(\vec{k}_i \cdot \vec{r}_j - \omega_i t)]$ for each incoming particle i , a factor $\exp[-i(\vec{k}_o \cdot \vec{r}_j - \omega_o t)]$ for each outgoing particle o , and a vertex factor proportional to a_{nm} .

The example can be generalized to an arbitrary number of sources and receivers (or, equivalently, an arbitrary number of incoming and outgoing particles). If we consider re-scatterings, we arrive at Feynman's sum over paths.

Let us mention two interesting points:

- The accuracy with which the conservation of energy-momentum is valid depends on the size of the interaction region and the time it takes for the interference to build up. The result of a “back-of-the envelope” calculation is the “uncertainty principle”.
- Energy-momentum is conserved in the reference frame in which the “crystal” is at rest. Since, in vacuum, energy-momentum transforms as a 4-vector, energy-momentum is also conserved in a reference frame with constant velocity with respect to the “crystal”.

In summary, we have gained some insight into why energy-momentum is conserved, and why interactions occur in *packets* satisfying the Planck and De Broglie relations.

5 Particle in a box

Consider a particle that collides elastically with the sides of a box.³ The box is accelerated in the z -direction by some external means. At time $t = 0$ the box is at rest (with respect to the inertial reference frame), and the particle collides elastically with one wall of the box. At time $t = \Delta t$ the particle collides with the opposite wall of the box, which is now moving in the z -direction with velocity $V = a \cdot \Delta t$. The change of p_z of the particle is less in the second collision than in the first collision due to the velocity V . The difference, calculated doing two Lorentz transformations, is

$$\Delta p = \frac{2V}{c^2} E \quad (20)$$

if $V/c \ll 1$. The change of momentum per unit time is called “force”. The time to go back and forth across the box is $2\Delta t$ if $V/c \ll 1$. Therefore the force is given by Newton's equation

$$F = M \cdot a \quad (21)$$

with Einstein's famous inertial mass

$$M = \frac{E}{c^2}. \quad (22)$$

This relation is valid for particles in the box of arbitrary velocity: the energy per particle is (8) if $m = 0$, or (9) if $m > 0$.

It is instructive to repeat this problem with waves, applying appropriate boundary conditions at the moving wall. The same result is obtained.

6 How does the proton acquire mass?

The proton is composed of two u quarks and one d quark tied together by gluons.⁴ The mass of the proton is much larger than the mass of the u or d quarks, so we neglect the quark masses. The gluon force becomes weak when the quarks are close together, and becomes strong when the quarks are separated by more than ≈ 2 fm. This “running coupling” of the gluons determines the radius of the proton (which is measured, by photon scattering, to be 1.2 fm).

We try the following crude model: the proton is composed of three massless quarks in a box.⁵ For simplicity we take the box to be a cube of sides d with the same volume as a sphere of radius 1.2 fm. We place each of the three quarks in the lowest energy state with $\lambda/2 = d$. Then we obtain an inertial mass of the proton of order ≈ 0.96 GeV/ c^2 .

Excited states are predicted and observed.⁶

7 Exploring the limit $m \rightarrow 0$.

From now on we generally set $c = \hbar = 1$. A massive free field ϕ satisfies the wave equation

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0, \quad (23)$$

which is equivalent to the dispersion relation (2). This differential equation can be written in integral form as

$$\phi(x) = \phi_0(x) - \int S_0(x-y) m^2 \phi(y) d^4 y, \quad (24)$$

where $\phi_0(x)$ is the general solution of

$$\partial_\mu \partial^\mu \phi_0 = 0, \quad (25)$$

and $S_0(x)$ is a particular solution of

$$\partial_\mu \partial^\mu S_0(x) = \delta^4(x). \quad (26)$$

A solution of (26) is

$$S_0(x) = -\frac{1}{(2\pi)^4} \int \frac{e^{-ikx}}{k^2} d^4 k. \quad (27)$$

The $d^4 k$ integral can be done (using the Cauchy theorem with the Feynman handling of the poles) and we obtain

³This model was inspired by [1].

⁴This crude toy model neglects quarks in the sea, and the energy of gluons.

⁵Equivalently, the proton may be viewed as a bubble in superconducting vacuum at pressure P filled with an ultra relativistic degenerate Fermi gas with 3 quarks. (?)

⁶It is best to use spherical coordinates to classify the excited states by their quantum numbers.

The dk^0 integral can be done (using the Cauchy theorem with the Feynman handling of the poles) and we obtain

$$S_0(x) = \frac{i}{16\pi^3} \int \frac{\exp[i(\vec{k} \cdot \vec{r} - |\vec{k}|t)]}{|\vec{k}|} d^3\vec{k} \quad \text{for } t > 0, \text{ and}$$

$$S_0(x) = \frac{i}{16\pi^3} \int \frac{\exp[i(\vec{k} \cdot \vec{r} + |\vec{k}|t)]}{|\vec{k}|} d^3\vec{k} \quad \text{for } t < 0. \quad (28)$$

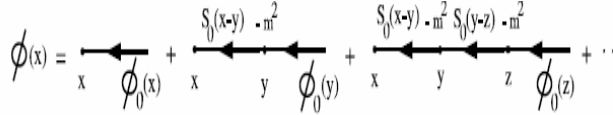


Figure 1: Feynman diagrams of a massless scalar particle acquiring mass by forward scattering on the vacuum.

The subscript 0 on $S_0(x)$ reminds us that this is the propagator of a zero mass field. Now we can treat m^2 as a perturbation, and solve (24) iteratively:

$$\phi(x) = \phi_0(x) + \int S_0(x-y)(-m^2)\phi_0(y)d^4y$$

$$+ \int S_0(x-y)(-m^2)S_0(y-z)(-m^2)\phi_0(z)d^4y d^4z + \dots \quad (29)$$

This series can be represented with Feynman diagrams as in Figure 1. Calculating the terms in the series we obtain

$$\exp[i(\vec{k} \cdot \vec{r} - \sqrt{m^2 + |\vec{k}|^2}t)] = \exp[i(\vec{k} \cdot \vec{r} - |\vec{k}|t)] \left(1 - \frac{i m^2}{2|\vec{k}|}t - \frac{1}{8} \frac{m^4}{|\vec{k}|^2}t^2 + \dots \right), \quad (30)$$

which is an expansion of the solution of (23) around the solution of (25) valid for $m^2/|\vec{k}|^2 \ll 1$.

The scatterings illustrated in Figure 1 conserve energy-momentum because the scattering amplitude $-m^2$ does not depend on space-time. Note that the propagation of a massive particle can be viewed as the propagation of a massless particle that forward scatters 0, 1, 2... times on the vacuum with amplitude $-m^2$. We will call this the “stepping stone” model of mass.

8 How does the electron acquire mass?

The Dirac field ψ has 4 components and carries a reducible representation of the proper Lorentz group. The irreducible components are Weyl-L and Weyl-R⁷ of dimension 2:

⁷In this article we will call “Weyl-L” (“Weyl-R”) the representation $(\frac{1}{2}, \frac{3}{2})$ ($(\frac{1}{2}, -\frac{3}{2})$) of the proper Lorentz group in the notation of [2]. In this notation, the scalar representation is (0,1), and the vector representation is (0,2).

$$\psi = \psi_L \oplus \psi_R, \quad \psi_L = \frac{1}{2}(1 - \gamma^5)\psi, \quad \psi_R = \frac{1}{2}(1 + \gamma^5)\psi. \quad (31)$$

The Dirac field carries irreducible representations of the complete (including space reflection) and general (including space and time reflections) Lorentz groups. In other words, proper Lorentz transformations do not mix ψ_L with ψ_R , but space or time reflections do. The Dirac equation for a free electron can be written as

$$-i\gamma^\mu \partial_\mu \psi_L = -m\psi_R,$$

$$-i\gamma^\mu \partial_\mu \psi_R = -m\psi_L, \quad (32)$$

where γ^μ are the Dirac matrices.

To try to understand how the particle acquires mass we proceed as follows. The differential equations (32) can be written in integral form:

$$\psi_L(x) = \psi_{L0}(x) + \int [-iS'_0(x-y)](-im)\psi_R(y)d^4y,$$

$$\psi_R(x) = \psi_{R0}(x) + \int [-iS'_0(x-y)](-im)\psi_L(y)d^4y, \quad (33)$$

where

$$-i\gamma^\mu \partial_\mu \psi_{L0} = 0, \quad -i\gamma^\mu \partial_\mu \psi_{R0} = 0, \quad (34)$$

and

$$-i\gamma^\mu \partial_\mu S'_0(x) = \delta^4(x). \quad (35)$$

Equations (33) can be iterated and we obtain the series

$$\psi_L(x) = \psi_{L0}(x) + \int [-iS'_0(x-y)](-im)\psi_{R0}(y)d^4y$$

$$+ \int [-iS'_0(x-y)](-im)[-iS'_0(y-z)](-im)\psi_{L0}(z)d^4y d^4z + \dots \quad (36)$$

and a similar series for $\psi_R(x)$. The third term on the right hand side of Equation (36) can be represented by the Feynman diagram in Figure 2. Note that the propagation of a massive electron can be viewed as the propagation of its massless Weyl components ψ_L and ψ_R that forward scatter 0, 1, 2... times on the vacuum with amplitude $-im$. Each scattering changes an incoming ψ_L component into an outgoing ψ_R component, and *vice-versa*.

The series (36) illustrates how an electron scatters on vacuum and acquires mass coupling the ψ_L and ψ_R components (which is also apparent in (32)). In the limit $m \rightarrow 0$ it is best to do the integral

$$\int [-iS'_0(x-y)] \cdot [-iS'_0(y-z)]d^4y = S_0(x-z), \quad (37)$$

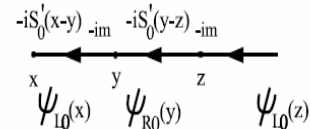


Figure 2: A Feynman diagram of a massless Dirac particle acquiring mass by forward scattering on the vacuum.

and recover the series (29).

As we shall see, the “stepping stone” picture presented in this Section may be significantly changed in the standard model when radiative corrections are taken into account.

9 Weyl fields in a box

Let us use a block-diagonal base such that the first two components of ψ are ψ_L , and the last two components of ψ are ψ_R . In this basis

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}, \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} -\sigma^0 & 0 \\ 0 & \sigma^0 \end{pmatrix}, \quad (38)$$

where σ^0 is the 2×2 unit matrix, and σ^k , with $k = 1, 2, 3$, are the Pauli matrices.

The solution of Equations (32) corresponding to a right handed electron propagating in the z -direction is

$$\psi \propto \begin{pmatrix} e^{-\phi/2} \\ 0 \\ e^{\phi/2} \\ 0 \end{pmatrix} \times e^{i(kz - \omega t)}, \quad (39)$$

where $\omega^2 - k^2 = m^2$, $\omega = m \cosh \phi$, $k = m \sinh \phi$. The solution corresponding to a left handed electron propagating in the $-z$ -direction is

$$\psi \propto \begin{pmatrix} e^{\phi/2} \\ 0 \\ e^{-\phi/2} \\ 0 \end{pmatrix} \times e^{i(-kz - \omega t)}. \quad (40)$$

The limit $m \rightarrow 0$, keeping $\omega \approx k$ finite, is singular: the ψ_L and ψ_R components become decoupled because $\phi \rightarrow \infty$.

Consider a box at rest of height l . A (massless) Weyl-R field propagates up along z . Its z -component of spin is $s_z = +\frac{1}{2}$. A (massless) Weyl-L field is reflected down. Its z -component of spin is $s_z = +\frac{1}{2}$. Note that angular momentum is conserved in the reflection. The box has no spin. We combine (39) and (40), take the limit $\phi \rightarrow \infty$, and change the notation $\omega \approx k \rightarrow m$. The field becomes

$$\propto \begin{pmatrix} \exp\{im(-z - t)\} \\ 0 \\ \exp\{im(z - t)\} \\ 0 \end{pmatrix} \quad (41)$$

where, in this block diagonal basis, the first two components are ψ_L propagating down, and the last two components are ψ_R propagating up.

A second observer (t', x', y', z') sees the box moving in the z -direction with velocity $v \equiv \tanh \phi$. The frequency of the up ward propagating ψ_R field is Doppler shifted to $\omega' = e^\phi m$, and the time it takes to traverse the height l of the box is dilated to $e^\phi l$. The frequency of the down ward propagating ψ_L field is Doppler shifted to $\omega' = e^{-\phi} m$, and the time it takes to traverse the height l of the box is shortened to $e^{-\phi} l$. So, the field as viewed by the second observer, is

$$\propto \begin{pmatrix} \exp\{-\frac{\phi}{2}\} \exp\{ie^{-\phi} m(-z' - t')\} \\ 0 \\ \exp\{\frac{\phi}{2}\} \exp\{ie^\phi m(z' - t')\} \\ 0 \end{pmatrix} \quad (42)$$

as in (39) and (40) in the zero mass limit. So we understand why the field transforms as it does!

Note that the probability of observing the particle moving up with velocity 1 is $e^\phi / (e^\phi + e^{-\phi})$, and the probability of observing the particle moving down with velocity -1 is $e^{-\phi} / (e^\phi + e^{-\phi})$, so the mean velocity of the particle is $\tanh \phi = v$ as expected. In the limit $v \rightarrow 1$, $\phi \rightarrow \infty$, and we are left with only the field ψ_R . The fields in the box acquire inertia with mass m , as we have explained in Section 5.

The model of this section may apply to a quark inside the proton. If it applies to an electron, then a modification of the Dirac equation is needed (since the Dirac equation only includes forward scatterings that conserve energy-momentum). Such a modification may be included in the standard model as we shall see shortly.

The ψ_L and ψ_R fields of this section satisfy the *massless* Dirac equation. Can the *massive* Dirac equation with mass m be applied to the box with

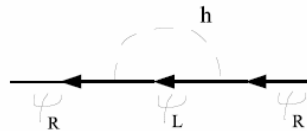


Figure 3: Feynman diagram of an electron in a “box”.

Weyl fields as a whole? If so, the box with massless fields ψ_L and ψ_R would be a description of an electron at a deeper level. This is similar to neutron diffraction by a crystal, described by a wave assigned to the neutron as a whole, in spite of the internal wheels and gears of the neutron. As another example, “electrons” and “holes” in a semiconductor can be treated as (quasi) particles in their own right with their (effective) masses, or, at a deeper level, as electron waves being Bragg-reflected back and forth by the periodic potential of the crystal. In fact, the model of standing “waves in a box” is similar to the standing electron waves in a crystal when the Bragg condition is satisfied.

10 What is the box made of?

From our preceding discussions on mass and inertia, it is crucial to conserve energy-momentum as the electron bounces back and forth across the box. The box must therefore carry a compensating momentum back and forth. So we identify the box with a field that carries energy-momentum. Since angular momentum is conserved in the reflection, we choose a scalar field.⁸ The Feynman diagram of the electron bouncing back and forth due to the interaction with this scalar field h is shown in Figure 3. We conclude that the Lagrangian should have terms proportional to

⁸A vector field may also be considered. In this case ψ_R bounces back as ψ_R , not as ψ_L . In the standard model the electron mass becomes renormalized by both the scalar Higgs field and by the gauge boson vector fields.

11 Bilinear forms of Weyl fields

In this Section we use the notation of reference [2] for the irreducible representations of the proper Lorentz group: $(\frac{1}{2}, -\frac{3}{2})$ is Weyl-R, $(\frac{1}{2}, \frac{3}{2})$ is Weyl-L, $(0, 1)$ is scalar, and $(0, 2)$ is vector. Every term in the Lagrangian is a scalar with respect to the proper Lorentz group. Consider the following direct products:

$$\left(\frac{1}{2}, \frac{3}{2}\right) \otimes \left(\frac{1}{2}, \frac{3}{2}\right) = (0, 1) \oplus (1, 2), \quad (44)$$

$$\left(\frac{1}{2}, -\frac{3}{2}\right) \otimes \left(\frac{1}{2}, -\frac{3}{2}\right) = (0, 1) \oplus (1, -2), \quad (45)$$

$$\left(\frac{1}{2}, \frac{3}{2}\right) \otimes \left(\frac{1}{2}, -\frac{3}{2}\right) = (0, 2). \quad (46)$$

Also, the Hermitian-conjugate of the matrices of $(\frac{1}{2}, -\frac{3}{2})$ carry the representation $(\frac{1}{2}, \frac{3}{2})$, and *vice versa*. So, scalar bilinear forms exist for ψ_L (ψ_R) that can give chargeless Weyl-L (Weyl-R) neutrinos a ‘‘Majorana mass’’. Similarly, the scalar bilinear form $\tilde{\psi}_R\psi_L + \tilde{\psi}_L\psi_R$ can give the Dirac field a ‘‘Dirac mass’’. Finally, the bilinear forms $\tilde{\psi}_R\gamma^\mu\psi_R$ and $\tilde{\psi}_L\gamma^\mu\psi_L$ are vectors, as needed for terms in the Lagrangian with first derivatives of the fields, so, for example, $\tilde{\psi}_R\gamma^\mu\partial_\mu\psi_R$ is a scalar.

12 The standard model

The standard model is illustrated in Figure 4. It contains Weyl-L fields (black dots), Weyl-R fields (white dots), and complex Higgs scalars (squares). The standard model has the following *global* symmetries:

- Invariance with respect to phase transformations $e^{iY_j\phi}$ (U_1 symmetry) independently for each field j within the rectangle ‘‘ U_1 ’’.
- Invariance with respect to SU_2 ‘‘rotations’’ independently among each pair of fields united by vertical bars inside the ‘‘ SU_2 ’’ rectangle. All fields within the rectangle ‘‘ SU_2 ’’ carry the representation 2 of SU_2 . Their complex conjugates carry the representation $\bar{2} = 2$ of SU_2 . All fields out of this rectangle are singlets of SU_2 .
- Invariance with respect to SU_3 transformations independently among each triplet of fields united by circles inside the ‘‘ SU_3 ’’ rectangle. All fields within the rectangle ‘‘ SU_3 ’’ carry the representation 3 of SU_3 . Their complex conjugates carry the representation $\bar{3}$ of SU_3 . All fields out of this rectangle are singlets of SU_3 .

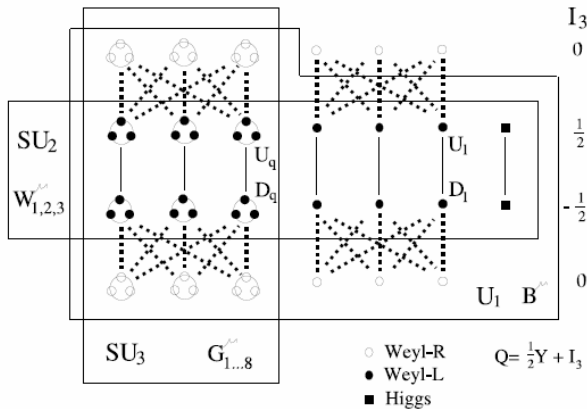


Figure 4: The elementary particles of the standard model.

So far the fields of the model are non-interacting. To create interactions, *some* of the *global* symmetries are promoted to *local* symmetries at each space-time point:⁹

- *one* U_1 transformation with a phase ϕ common to all fields within the rectangle ‘‘ U_1 ’’ (each of these fields has its own Y_j quantum number),
- *one* SU_2 transformation with the three generators of SU_2 common to all fields within the rectangle ‘‘ SU_2 ’’, and
- *one* SU_3 transformation with the eight generators of SU_3 common to all fields within the rectangle ‘‘ SU_3 ’’.

To achieve the invariance of the model with respect to these *local* (or ‘‘gauge’’) symmetries, it is necessary to replace ordinary derivatives ∂_μ by covariant derivatives $D_\mu = \partial_\mu + igY_j B_\mu + ig\frac{\tau}{2} \cdot \vec{W}_\mu + ig_3 T_a G_\mu^a$. These covariant derivatives have one vector gauge field for every generator of the symmetry group: B^μ for the generator of U_1 , \vec{W}^μ for the three generators of SU_2 , and $G_1^\mu, G_2^\mu, \dots, G_8^\mu$ for the 8 generators of SU_3 . There are 3 coupling constants: g' for U_1 , g for SU_2 , and g_3 for SU_3 . Due to the gauge (and Higgs) fields, the Lagrangian becomes non-linear, the fields interact, and *particles* emerge!

The invariance of the Lagrangian with respect to the transformation U_1 implies the conservation of the quantum number Y at every vertex of a Feynman diagram. The invariance with respect to SU_2 implies the conservation of I_3 . The electric charge is $Q = \frac{1}{2}Y + I_3$. The invariance with respect to SU_3 implies the conservation of 2 ‘‘color’’ quantum numbers (because two of the Gell-Mann SU_3 matrices are diagonal).

The symmetries of the standard model prevent adding mass terms ‘‘by hand’’ to the Lagrangian, except to the Higgs field.¹⁰ All other particles acquire mass if the ground state of the Higgs field acquires a vacuum expectation value and ‘‘hides’’ the SU_2 symmetry. Then the interactions with the vacuum expectation value of the Higgs field pairs up Weyl-L and Weyl-R fields giving mass to the resulting Dirac fields. The Weyl fields that are paired up are ‘‘rotated’’ among the three families, as indicated in Figure 4 by the matrices U_q, D_q, U_l, D_l , and by the dotted lines. The unitary Cabibbo-Kobayashi-Maskawa matrix is $U_q^\dagger D_q$. There is a similar matrix $U_l^\dagger D_l$ mixing families in the lepton sector. No other rotations among families have experimental consequences. The mass of the Dirac fields can not be calculated in the standard model because their ‘‘Yukawa coupling’’ to the Higgs are put in by hand, and also because the masses become renormalized by radiative corrections. So, in the end, *the experimental masses of quarks and leptons must be put in by hand at each order in perturbation theory: at least for the time being, they are not calculable even in principle!*

⁹Model builders: note that only *some* of the global symmetries are gauged.

The real magic of the Higgs mechanism is how it gives mass to the W^\pm and Z bosons, leaving the photon massless. The vacuum expectation value v of the Higgs field gives the W^\pm a “tree level” mass $vg/2$, and the Z a “tree level” mass $v(g^2 + g'^2)^{1/2}/2$. The complex Higgs SU_2 doublet has four amplitudes. Three of them can be “gauged away” and become the longitudinal polarizations needed by W^\pm and Z to become massive. Diagrams that contain these longitudinal polarizations diverge. These divergences are canceled, order by order in the perturbation expansion, by the only remaining amplitude h of the Higgs field. It is magic!

13 Mass revisited

The term in the standard model Lagrangian that gives mass to the electron is [3]

$$-G_e \frac{v+h}{\sqrt{2}} \left(\tilde{\psi}_R \psi_L + \tilde{\psi}_L \psi_R \right), \quad (47)$$

where G_e is a dimensionless “Yukawa coupling” put in by hand. So the vacuum expectation value v of the Higgs field gives the electron a “tree level” mass $m_e = G_e v / \sqrt{2}$. Note in (47) that the Lagrangian indeed contains a term (43). So the standard model contains loops as shown in Figure 3. Summing a series with $0, 1, 2 \dots$ loops we obtain a propagator with the renormalized mass

$$m_e = \frac{G_e}{\sqrt{2}} \left(v + \frac{G_e}{\sqrt{2}} A + \dots \right), \quad (48)$$

where the factor A has the form

$$A = -i \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k_\mu k^\mu - m_h^2} \frac{i}{\gamma^\mu (p_\mu + k_\mu) - \frac{G_e v}{\sqrt{2}}}. \quad (49)$$

The contributions of higher order primitive graphs are indicated by dots in (48).

In the standard model, the integral (49) diverges (apparently) linearly as the regularization is turned off. In this case, the second term in the parenthesis of (48) diverges, keeping $m_e = \frac{1}{2} G_e^2 A$ finite. However, the standard model is not expected to be the final theory of nature. New physics will emerge at the Planck scale, or grand unified scale, or even lower, and effectively cuts off the integral (49).

We then have two alternatives. In the first alternative, the term with v dominates in (48), and the electron acquires mass by forward scattering on the vacuum as explained in Section 8. This is the “stepping stone” model. In the second alternative, the terms $\frac{G_e}{\sqrt{2}} A + \dots$ dominate, and we obtain the model of Weyl-L and Weyl-R fields reflecting back and forth in a box made of the Higgs field as described in Sections 9 and 10. This is the “waves in a box” model. The two alternatives are shown schematically in Figure 5. Which alternative has nature chosen? Or has nature chosen a combination of both?

¹⁰Mass terms that break the symmetries of the theory destroy its renormalizability, and no predictions become possible (unless new physics cuts off the diverging integrals).

14 Experimental consequences

The proton can be viewed as ultrarelativistic quarks in a box made of gluons. The radius of the box has been measured, for example, with $\gamma p \rightarrow \gamma p$

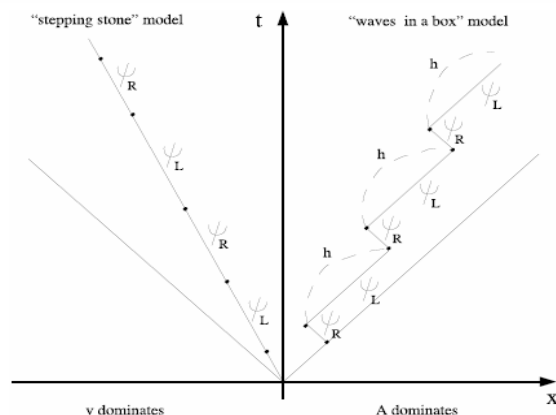


Figure 5: Two alternatives to acquire mass.

scattering. Note that the proton acquires mass independently of the vacuum expectation value of the Higgs field, and this mass can be calculated with lattice quantum chromodynamics.

Note that scattering with momentum transfer $\sqrt{|t|} \lesssim 4.6/R$ probes the radius R of the box as a whole, while scattering with $\sqrt{|t|} \gtrsim 4.6/R$ probes its constituents.

The case of the electron is more uncertain. Apparently, the electron acquires mass by the “stepping stone” mechanism. If the electron can be considered as “waves in a box”, then the box has size $\approx \hbar/(mc)$, the field h would have to be nearly massless, and the size of the box would cause the Darwin shift of the energy levels of hydrogen.

Many questions remain. The reason for an effective, low energy, theory to be renormalizable is to decouple it from the high energy theory. But, in the standard model, the Higgs sector does not decouple. What are the consequences on e^+e^- scattering of Feynman graphs with Higgs in a loop? If new physics cuts off otherwise divergent integrals such as (49), what are the experimental consequences? Can we constrain new physics this way?

15 Conclusions

I have tried to understand the origin of mass and have failed miserably (which may be a step in the right direction). Nature, as we currently understand it, is described by a perturbation expansion of Feynman diagrams. We subscribe to the belief that “the diagrams contain more truth than the underlying formalism” [’t Hooft and Veltman (1973)]. We can not calculate the electron mass: we must introduce a “Yukawa coupling” by hand *at each order in the perturbation expansion*. So, at our present level of understanding, *it appears hopeless to calculate the masses of quarks and leptons*: we must take their values from experiment. On the other hand, we can calculate the mass of baryons and mesons using lattice quantum chromodynamics.

Early in this paper we arrived at a beautiful insight. We considered *classical waves* that interact. We found that the condition that the waves add up in phase at the receiver has the form

$$\begin{aligned} n\omega_1 + m\omega_2 + \dots &= 0, \\ n\vec{k}_1 + m\vec{k}_2 + \dots &= 0, \end{aligned} \quad (50)$$

where the powers $n, m \dots$ of the Taylor expansion of the non-linear interaction, are integers. Note that the dispersion occurs *as if the waves were composed of discrete packets* of frequency-wavevector (ω_j, \vec{k}_j) . The packets are called “particles”, and $(E_j, \vec{p}_j) \equiv (\hbar\omega_j, \hbar\vec{k}_j)$ is the “energy-momentum” of particle j . According to (50), energy-momentum is conserved (with a precision determined by the uncertainty principle, i.e. by the size of the region of space-time over which the interference builds up). Note that the concept of particle, and the Planck and De Broglie relations, have emerged from the interaction of *classical* waves!

In semiconductor physics we can consider quasi-particles called “electrons” and “holes” with their effective masses, or, at a deeper level, we can consider electrons being Bragg-reflected back and forth by the periodic potential of the crystal. Similarly, we can consider the massive Dirac electron, or, at a deeper level, we can perhaps consider massless Weyl-L and Weyl-R fields being reflected back and forth by emitting and absorbing virtual scalar Higgs particles. Similarly, we can perhaps consider massive gauge bosons as massless bosons being reflected back and forth by emitting and absorbing virtual scalar Higgs particles. These reflections give the gauge bosons their longitudinal polarization (as reflections in a waveguide give the photon a mass and a longitudinal polarization). Which model best describes nature, “waves in a box” or “stepping stone”, or a combination of both, will depend on which term in (48) dominates.

The perturbation expansion of Feynman diagrams has all imaginable diseases. We may hope that an expansion is possible around a less singular starting point. For some applications it may prove convenient to replace massive fermions and bosons by the corresponding “waves in a box”.

In the process of trying to understand mass, I have arrived at the following view of the standard model. Quarks and leptons are not elementary at all. An electron is a messy composite of a superposition of three Weyl-L fields, a Weyl-R field and a Higgs field coupled together by a Yukawa coupling, and all of them are coupled to gauge fields by gauge couplings, which also interact with themselves and everything else. The energy of this mess has inertia, as all energy does, and it is that mess that determines the mass of the electron. So the next level in our understanding of Nature, from atoms to nuclei and electrons, to baryons and mesons, to quarks and leptons, are the dots shown in Figure 4, and these have no mass (except the Higgs?).

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- [3] Claude Itzykson and Jean-Bernard Zuber, “Quantum Field Theory”, McGraw-Hill Book Company (1980).