Application of a Discontinuous Controller with Chattering Attenuation to Unicycle Mobile Robots

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Abstract— **We developed a discontinuous controller to stabilize the movement of a unicycle mobile robot around a desired trajectory via backstepping methodology. The main contribution of the paper is that chattering signal, produced by the imprecision of the discontinuous terms, vanished thanks to an adaptive gain that reduces the chattering level once the position errors were zero. A stability analysis of the closed-loop dynamic system in question was performed within the framework of Lyapunov functions. Performance issues related to the discontinuous controllers are illustrated in numerical simulations applied to a unicycle mobile robot.**

I. Introduction

A. Overview

Mobile robots have attracted considerable interest among researchers in the robotics and control community because they posses nonholonomic properties caused by non-integrable differential constraints. This feature makes them interesting because the motion of nonholonomic mechanical systems are constrained by its own kinematics therefore control laws are not straightforwardly derivable. A suitable choice to solve the tracking control problem of mobile robots, including both kinematic and Euler-Lagrange model, is using backstepping methodology. Recent developments include the introduction of an adaptive tracking controller based on a backstepping approach [6]; and a dynamical extension that makes possible the integration of kinematic and torque controllers for a nonholonomic mobile robot proposed by Fierro and Lewis [5]. The backstepping approach consists in two steps: 1) find the desired velocities such that stabilize the kinematic model and 2) find a control law to ensure that real velocities converge to the desired velocities.

Variable structure control (VSC) is a recognized method to solve the tracking control problem of nonholonomic robot systems via backstepping because of the required finite-time stabilization of its linear and angular velocities (see, *e.g.* [10], [11], [16]). However, the main drawback of the VSC controllers is the infinite fast switching of the input control called chattering that appears when trajectories reach the sliding manifold (or the equilibrium point) and it is mainly caused by the imprecision of the discontinuous terms and that excite non desirable dynamics as consequence ([4], [15]).

Few but significant results have appeared in research papers dealing with the chattering problem: Parra-Vega *et al.* [14], for example, showed that an adaptive and nonadaptive cases of variable structure robot control undergoes chattering attenuation. Bartolini *et al.* [1] demonstrated that it is possible the chattering elimination by generating a second-order sliding motion by using as control the first derivative of the control instead of the actual control. Another alternative used in control applications is to replace the signum function by its smooth version (*e.g.* tanh, sigmoid function, among others).

Further motivation of the use of VSC control, in the backstepping approach, is the finite-time convergence of the output to the equilibrium point. Several analysis of finite-time convergence can been found in literature (see *e.g.* [2], [10], [13], [16] and references therein).

B. Contribution and organization of the paper

The contributions of this paper are twofold: Chattering attenuation and finite-time stabilization of the proposed algorithm. To overcome the problems that results from chattering, we propose the addition of an extra dynamic that auto-tunes the gain value that multiply the discontinuous term so that its gain value decreases once the trajectory of the system reaches the desired ones, thereby decreasing the chattering level. The mobile robots have actuated wheels, so the real control input is torque and we included this consideration when developing our controller. For this reason, we sought a control law such that ensure finite-time stabilization of velocities, thus stabilizing the robot trajectories.

This paper is organized as follows: Section II presents the kinematic and dynamic model of the unicycle mobile robot; Section III introduces the discontinuous tracking controller with chattering attenuation along with its stability analysis, a prove of finite-time convergence and a comparative study for a first-order discontinuous controller; Section IV provides a simulation study for the unicycle mobile robot using the controller described in Section III; and Section V presents the conclusions of this study.

C. Notation

The following definition will be used throughout the paper. The norm $\|x\|_2$, with $x \in \mathbb{R}^n$, denotes the Euclidean norm and $\|x\|_1 = |x_1| + \ldots + |x_n|$ stands for the sum norm. The minimum and maximum eigenvalue of a matrix $A \in \mathbb{R}^{n\times n}$ is denoted by $\lambda_{\min}\{A\}$ and $\lambda_{\max}\{A\}$, respectively. The vector $\text{sign}(x)$ is given by $\text{sign}(x) = [\text{sign}(x_1), \ldots, \text{sign}(x_n)]^T$ where the signum function is defined as

$$\text{sign}(y) = \begin{cases} 1 & \text{if } y > 0, \\ \{-1, 1\} & \text{if } y = 0 \\ -1 & \text{if } y < 0, \end{cases} \quad \forall\, y \in \mathbb{R}.$$

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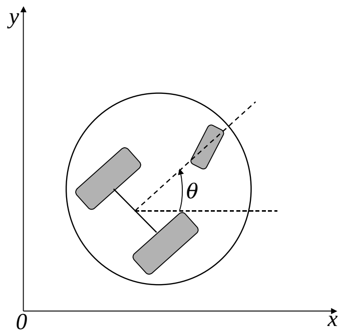

Fig. 1. Unicycle mobile robot.

II. Dynamic model

In this paper, we propose a discontinuous tracking controller with chattering attenuation for a unicycle mobile robot governed by the following equation of motion ([9]):

$$\dot{q} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (1)$$

$$M(q)\dot{v} + C(q,\dot{q})v + Dv = u \quad (2)$$

where $q = [x, y, \theta]^T$ is the vector of generalized coordinates; $v = [v_1, v_2]^T$ is the vector of velocities; u is the vector of applied torques; $M(q)$ is the positive-definite inertia matrix; $C(q,\dot{q})$ is the vector of centripetal and Coriolis forces; and D is the diagonal positive-definite damping matrix. Equation (1) represents the kinematics of the system, where (x, y) are the Cartesian coordinates; θ is the angle between the heading direction and the x-axis; and v_1 and v_2 are the linear and angular velocities, respectively (see Fig. 1). Furthermore, the system (1)-(2) has a nonholonomic constraint described by

$$\dot{y}\cos\theta - \dot{x}\sin\theta = 0. \quad (3)$$

This constraint means that a no-slip condition is imposed on the wheels, so the robot can not move sideways [12].

III. Discontinuous tracking controller

The control objective is established formally as follows: Given a bounded and continuously differentiable desired trajectory and orientation for the mobile robot $q_d(t)$, we must design a discontinuous control law u such that the positions $q(t)$ reach the desired reference position $q_d(t)$ asymptotically, that is,

$$\lim_{t\to\infty} \|q_d(t) - q(t)\| = 0. \quad (4)$$

To achieve the objective control (4), we followed the backstepping methodology ([8]) consisting of the following two steps:

1) We must find an ideal velocity vector $v_r = v$ such that the kinematic system (1) be asymptotically stable.
2) A control input u must be found such that

$$\lim_{t\to t_s} \|v_r - v\| = 0 \quad (5)$$

where $t_s < \infty$ is the reachability time. Real mobile robots have actuated wheels, so the control input is u.

A. Control of the kinematic model

To solve the tracking control problem for the kinematic model, we followed the procedures proposed by Fierro and Lewis [5] and Lee *et. al.* [9]. Suppose the reference trajectory $q_d(t)$ satisfies

$$\dot{q}_d = \begin{bmatrix} \cos\theta_d & 0 \\ \sin\theta_d & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_{1d} \\ v_{2d} \end{bmatrix}. \quad (6)$$

Using the robot local frame, the error coordinates can be defined as

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_d - x \\ y_d - y \\ \theta_d - \theta \end{bmatrix},$$

therefore, the tracking error model is given by

$$\begin{cases} \dot{e}_1 = v_2 e_2 - v_1 + v_{d1}\cos e_3 \\ \dot{e}_2 = -v_2 e_1 + v_{d1}\sin e_3 \\ \dot{e}_3 = v_{d2} - v_2. \end{cases} \quad (7)$$

Theorem 1 ([5]): Let the tracking error system (7) be driven by the control law

$$v_{r1} = v_{d1}\cos e_3 + k_1 e_1 \quad (8)$$

$$v_{r2} = v_{d2} + v_{d1}k_2 e_2 + k_3 \sin e_3 \quad (9)$$

where k_1, k_2, and k_3 are strictly positive constants. If $v_1 = v_{r1}$ and $v_2 = v_{r2}$, then the origin of the closed-loop system [(7)-(9)] is asymptotically stable.

Proof: Proof of theorem 1 can be found in [5]. ■

B. Computed torque feedback controller

Before show the main result, we proceed to derive a discontinuous control law to compute the required torque for the actual unicycle mobile robot governed by (2).

In theorem 2, we propose a discontinuous control law to force the real velocities to that required in equations (8)-(9) of theorem 1 to satisfy the control objective (4), that is

$$\lim_{t\to\infty} \|v_r - v\| = 0. \quad (10)$$

Theorem 2: Let the system (2) be driven by the control law

$$u = M(q)[\dot{v}_r + k_s \text{sign}(\sigma) + k_p\sigma] + C(q,\dot{q})v + Dv \quad (11)$$

where k_s and k_p are diagonal positive-definite matrices and the output $\sigma = v - v_r$ is the translational and rotational velocity error vector. Then v tends to v_r in finite-time.

Proof: We look for a control law such that σ tends to 0 in finite-time. First, we prove asymptotic stability of the closed-loop system. The time derivative of the output vector σ is given by:

$$\dot{\sigma} = \dot{v}_r - \dot{v}, \tag{12}$$

and substituting (2) into (12), it follows that

$$\dot{\sigma} = \dot{v}_r - M^{-1}(q)\left[u - C(q,\dot{q})v - Dv\right].$$

With the control input given in (11), we have

$$\dot{\sigma} = -k_s \text{sign}(\sigma) - k_p\sigma. \tag{13}$$

Using the Lyapunov function candidate

$$V = \frac{1}{2}\sigma^T\sigma,$$

the time derivative along the trajectories of (13) results in $\dot{V} = -\text{sign}(\sigma)^T k_s\sigma - \sigma^T k_p \sigma \leq -\lambda_{\min}\{k_s\}\|\sigma\|_1 < 0$. Thus, we can conclude asymptotic stability by invoking theorem 4.1 from Khalil ([7, p. 114]).

Apart from this, inequality $-\|\sigma\|_1 \leq -\|\sigma\|_2$ allow us to bound from above the time derivative of V as follows

$$\dot{V} \leq -2\lambda_{\min}\{k_s\}\sqrt{V}. \tag{14}$$

Integrating equation (14) implies that the time taken to reach the equilibrium point denoted by t_s satisfies

$$t_s \leq \frac{\sqrt{V(0)}}{\lambda_{\min}\{k_s\}},$$

also concluding finite-time reachability. ■

C. Chattering amplitude attenuation

The controller proposed in (11), possess a discontinuous term that persist even when the position error is zero, producing constant chattering for all time. Chattering is a drawback of sliding mode control caused mainly by unmodeled dynamics that exist along the with the principal dynamics of the plant. Therefore, it is desirable to design a controller with the same robustness properties possessed by discontinuous controllers, but with a chattering that vanishes once the position error is zero. To introduce our design of adjustable amplitude chattering control, consider the following differential equation:

$$\dot{\delta} = -k_\beta\delta + k_r|\text{tanh}(\sigma)|, \quad \delta \in \mathbb{R}^2, \tag{15}$$

where k_β and k_r are 2×2 positive definite matrices and

$$|\tanh\sigma| = \begin{bmatrix} |\tanh\sigma_1| \\ |\tanh\sigma_2| \end{bmatrix}.$$

Theorem 3: Let be the system (2) be driven by the control law

$$u = M(q)[\dot{v}_r + \Delta k_s \text{sign}(\sigma) + k_p\sigma] + C(q,\dot{q})v + Dv \tag{16}$$

where k_s and k_p are diagonal positive-definite matrices; $\sigma = v - v_r$ is the translational and rotational velocity error vector; and $\Delta = \text{diag}\{\delta_i\}$, δ_i is the ith-element of δ obtained of system (15). Then, v tends to v_r in finite-time.

Proof: We split the proof in two parts: asymptotic stability and finite-time convergence.

1) With the control input given in (15)-(16), we have

$$\dot{\sigma} = -\Delta k_s \text{sign}(\sigma) - k_p\sigma. \tag{17}$$

To conclude asymptotic stability, consider the following Lyapunov function candidate

$$V = \frac{1}{2}\sigma^T\sigma + \frac{1}{2}\delta^T\delta.$$

The time derivative of V along the trajectories of (17) results in

$$\begin{aligned}\dot{V} = \sigma^T\dot{\sigma} + \delta^T\dot{\delta} = \sigma^T\left[-\Delta k_s \text{sign}(\sigma) - k_p\sigma\right] \\ +\delta^T\left[-k_\beta\delta + k_r|\text{tanh}(\sigma)|\right].\end{aligned}$$

Employing the well-known inequalities

$$\|\tanh x\|_2 \leq \alpha_1\|x\|_2, \quad \alpha_1 > 0, \tag{18}$$

$$2\|g\|\|h\| \leq \|g\|^2 + \|h\|^2, \quad g, h \in \mathbb{R}^n, \tag{19}$$

yields

$$\begin{aligned}\dot{V} &\leq [\lambda_{\max}\{k_s\} + \alpha_1\lambda_{\max}\{k_r\}]\|\sigma\|_2\|\delta\|_2 \\ &\quad -\lambda_{\min}\{k_p\}\|\sigma\|_2^2 - \lambda_{\min}\{k_\beta\}\|\delta\|_2^2 \\ &= -\begin{bmatrix}\|\sigma\|_2 \\ \|\delta\|_2\end{bmatrix}^T \mathcal{Q}\begin{bmatrix}\|\sigma\|_2 \\ \|\delta\|_2\end{bmatrix},\end{aligned}$$

with

$$\mathcal{Q} = \begin{bmatrix}\lambda_{\min}\{k_p\} - H & 0 \\ 0 & \lambda_{\min}\{k_\beta\} - H\end{bmatrix}$$

where $H = (1/2)(\lambda_{\max}\{k_s\} + \alpha_1\lambda_{\max}\{k_r\})$. To guarantee that $\dot{V}$ be negative definite we want to satisfy

$$\lambda_{\min}\{k_p\} > H, \tag{20}$$

$$\lambda_{\min}\{k_\beta\} > H, \tag{21}$$

thus concluding asymptotic stability of the origin by invoking theorem 4.1 from Khalil ([7, p. 114]).

2) To complete the proof, it remains to justify finite-time convergence of the output $\sigma(t)$ to the origin. To this end, we assume that the dynamic (17) is faster than (15). Now, consider the auxiliary function

$$W(\sigma) = \frac{1}{2}\sigma^T\sigma.$$

The time derivative of $W(\sigma)$ along the solution of (17) and taking into account from (15) that $\delta(t) > 0$, with

$\delta(0) > 0$, for all time, one gets that

$$\dot{W} = -\lambda_{\min}\{k_s\}\|\delta\||\sigma| - \lambda_{\min}\{k_p\}\|\sigma\|^2.$$

By virtue of asymptotical stability, above concluded, we have that $\|\delta\| \leq M$, with $M > 0$, therefore

$$\dot{W} \leq -\lambda_{\min}\{k_s\}M|\sigma| = -2\lambda_{\min}\{k_s\}M\sqrt{W}.$$

By following the procedure of subsection III-C, we obtain

$$t_s \leq \frac{\sqrt{W(0)}}{\lambda_{\min}\{k_s\}M},$$

thus concluding finite-time reachability.

■

Proof of finite-time stability in theorems 2 and 3 can be corroborated by the recent results obtained by Levant [10] who constructed high-order finite-convergence sliding mode controllers for exact tracking of the trajectories; and by Orlov [13] that demonstrate finite-time convergence of homogeneous systems with right-hand discontinuous sides.

IV. Simulation Results

We generated simulations to study the performance of the controllers. The desired inputs v_{d1} and v_{d2} were chosen as follows ([6]):

$$\begin{cases} 0 \leq t < 5: \; v_{d1} = 0.25(1 - \cos\frac{\pi t}{5}) \\ \qquad\qquad v_{d2} = -v_{d1}/1.5 \end{cases}$$

where $t \in \mathbb{R}$ is the time in seconds. In the simulations, performed with SIMNON®, the motion of the unicycle mobile robot was governed by (1)-(2), where

$$M(q) = \begin{bmatrix} 0.3749 & -0.0202 \\ -0.0202 & 0.3739 \end{bmatrix},$$

$$C(q,\dot{q}) = \begin{bmatrix} 0 & 0.1350\dot{\theta} \\ -0.1350\dot{\theta} & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix},$$

were taken from [3]. The position and velocity initial conditions were set to $q(0) = 0 \in \mathbb{R}^3$, $v(0) = 0 \in \mathbb{R}^2$ and $\delta(0) = 0.1 \in \mathbb{R}^2$ in all simulations. The control gains were selected as follows:

$$\begin{aligned} k &= (k_1, k_2, k_3) = (50, 200, 50), \\ k_p &= \text{diag}\{30, 30\}, \\ k_s &= \text{diag}\{5, 5\}, \\ k_\beta &= \text{diag}\{10, 5\}, \quad \text{and} \\ k_r &= \text{diag}\{1, 1\}. \end{aligned}$$

The resulting position errors and input torques of the closed-loop system [(1), (2), (11)] are depicted in Figs. 2 and 3, respectively. In Fig. 2 we show that position errors converge to zero for the closed-loop system [(1), (2), (11)], whereas the input torques (Fig. 3) presents high frequency oscillations (chattering). In Fig. 4 we also show that position errors converge to zero for the closed-loop system [(1), (2), (15), (16)] but the chattering level decrease asymptotically as can be seen in Fig. 5. Also note that convergence of trajectories are faster with the proposed algorithm. Finally, Fig. 6 presents the time evolution of δ.

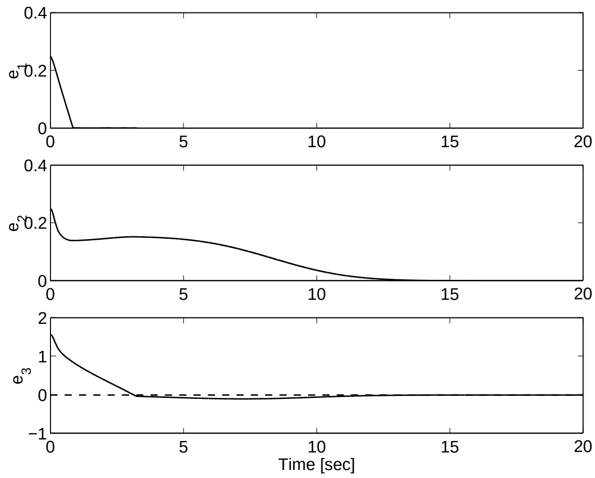

Fig. 2. Position errors.

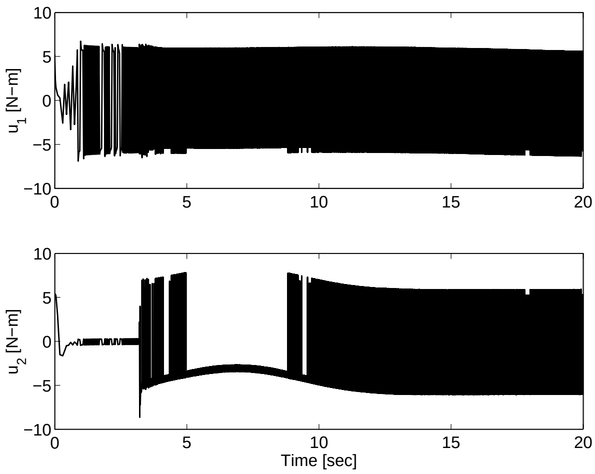

Fig. 3. Control input.

V. Conclusions

We have developed a discontinuous controller governed by a differential equation with a nonholonomic constraint that is applicable to mobile robots. To derive the controller, we used the backstepping methods and our newly proposed dynamic: an adaptive gain that attenuate the chattering signal inherent to variable structure systems without losing the robustness of the system. The stability analysis was developed within the Lyapunov function framework. Simulation studies showed the controller to be effective. This type of controller is being implemented in a prototype of an autonomous aircraft parking guidance currently.

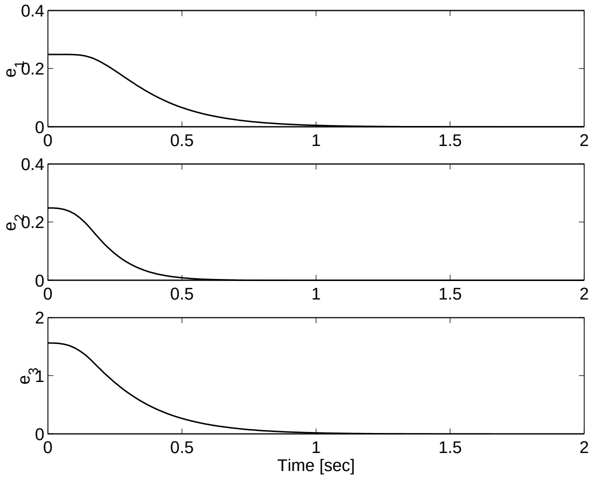

Fig. 4. Position errors using controller with chattering attenuator.

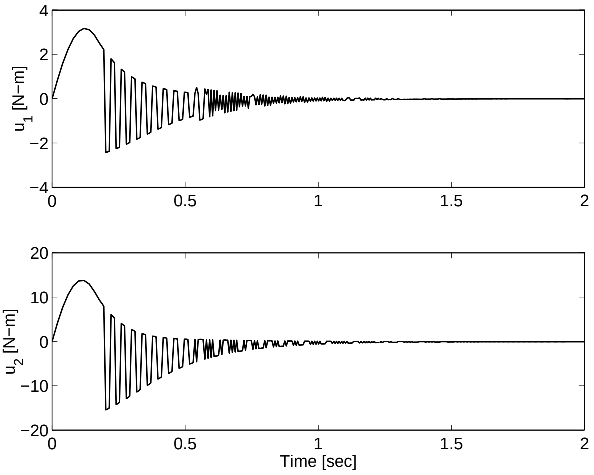

Fig. 5. Control input using controller with chattering attenuator.

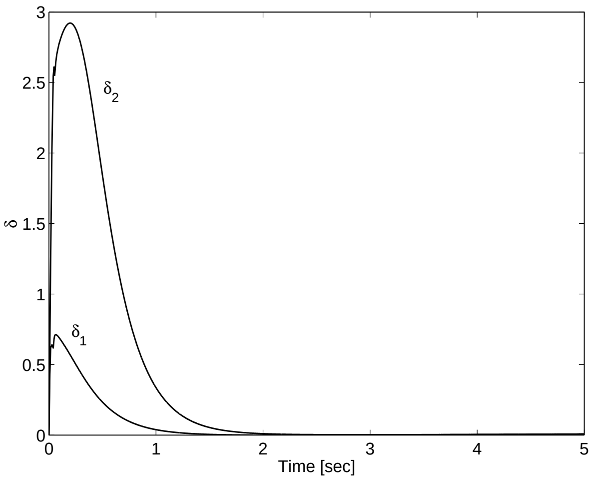

Fig. 6. Time response of δ.

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