Sliding mode control with linear matrix inequalities using only output information

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***Abstract*— This paper presents a *sliding mode control using only output information*. The robustness conditions come from a linear matrix inequality (LMI). This LMI permits to use output feedback to stabilize the system with matched and unmatched disturbances and uncertainties. The controller synthetized consists of two components. One component of linear output feedback regulates the system and rejects the unmatched disturbance. Other components of nonlinear output function provide robustness to the overall system. Robustness conditions are given. Simulations results using Simulink Matlab to simulate several nonlinear systems are presented.**

***Index Terms*— Linear Matrix Inequality, Sliding Mode Control, robustness control.**

I. Introduction

Sliding mode control is one the most important approaches to control nonlinear systems with uncertainties and disturbances. The power and electromechanical systems, and systems with high number of states and nonlinearities like friction, backlash, etc, are appropriate to apply this technique. This approach could give robustness to the system [1]. Robustness could decrease if only some states of the system are used to feedback. The answer of the question: can a linear system be controlled using only output feedback?. is very important the answer can be given in terms of feasibility of a LMI [2]. A linear system with linear output feedback can be posed how a LMI [3]. If this LMI is feasible the output feedback gain can be find and the system can be controlled.

There are several algoritms and software to solve feasibility problems of some LMIs (i.e. Matlab LMI toolbox and LMI tool from SCILAB) [4], [5]. These programs are used to calculate the linear output feedback controller gain and to obtain a positive defined matrix that permit to prove the robustness of the controlled system. Some nonlinear systems with uncertainties and disturbances can be written as the addition of a known linear system and an unknown function of disturbances and uncertainties [6]. Some information about this function is required (i.e. highest value and its limits). This function can be divided in two parts: one matched and one unmatched. The matched one can be rejected easily. On these systems sliding mode and linear output feedback controllers can be designed. Sliding mode controller permits to provide robustness to the system, rejecting the matched disturbances. The linear output feedback controller can reject, sometimes, the unmatched disturbances and permits to regulate the system.

II. Problem Statement

Consider the nonlinear system described by:

$$\begin{array}{l}\dot{X}(t) = Ax(t) + BU(t) + \Psi(t, X, U)\\ Y(t) = CX(t)\end{array} \tag{1}$$

$X(t) \in R^n$ is the state vector
$U(t) \in R^m$ is the input vector
$Y(t) \in R^p$ is the output vector
$A \in R^{n*n},\ B \in R^{n*m},\ C \in R^{p*n}$ are know matrices with $m \le p < n$; B and C are full rank.

$$\Psi(t, X, U) = \left[\begin{array}{l}\Psi_1(t, X)\\ \Psi_2(t, X, U)\end{array}\right] \tag{2}$$

$$\Psi_2(t, X, U) = \left[\begin{array}{l}\phi_1(t, X)\\ \phi_2(t, X, U)\end{array}\right] \tag{3}$$

About this functions only know:

$$\parallel \Psi_1(t, X) \parallel \le \Psi_{10} \parallel X(t) \parallel \tag{4}$$

$$\parallel \phi_1(t, X) \parallel \le \phi_{10} \parallel X(t) \parallel \tag{5}$$

$$\parallel \psi_2(t, X, U) \parallel \le \gamma_0 \parallel U(t) \parallel + \alpha(t, Y) \parallel \tag{6}$$

Where $\phi_1(t, X) \in R^{n-m}$; $\phi_2(t, X, U) \in R^m$, Ψ_{10}, ϕ_{10}, $0 \le \gamma_0 < 1$ are known positives constants and $\alpha_0(t, Y)$ is know positive function.

The system is in regular form [6] with:

$$\begin{array}{l} B = \left[\begin{array}{c}0\\ B_2\end{array}\right]\\ \\ B_2 = \left[\begin{array}{c}0\\ D_2\end{array}\right]\end{array} \tag{7}$$

Where $B_2 \in R^{p*m}$, $D_2 \in R^{m*m}$ and D_2 is non singular. And

$$C = [C_1\ C_2] \tag{8}$$

Where $C_1 \in R^{p*(n-p)}$, $C_2 \in R^{p*p}$

If this is not the case, the system can be written in this form using linear transformations.

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The system is transformed into a new state:

$$W(t) = \begin{bmatrix} X_1(t) \\ Y(t) \end{bmatrix} = TX(t) \tag{9}$$

Where $X_1(t) \in R^{n-p}$, $Y(t) \in R^p$, with

$$T = \begin{bmatrix} I & 0 \\ C_1 & C_2 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} I & 0 \\ -C_2^{-1}C_1 & C_2^{-1} \end{bmatrix}$$

The new system is:

$$\dot{W}(t) = \overline{A}W(t) + \overline{B}U(t) + \overline{\Omega}(t, W, U) \tag{10}$$

With

$$\begin{array}{l} \overline{A} = T^{-1}\ AT = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix} \\ \overline{A}_{11} = A_{11} - A_{12}C_2^{-1}C_1 \\ \overline{A}_{12} = A_{12}C_2^{-1} \\ \overline{A}_{21} = C_1A_{11} + C_2A_{21} - C_1A_{12}C_2^{-1}C_1 - C_2A_{22}C_2^{-1}C_1 \\ \overline{A}_{22} = C_1A_{12}C_2^{-1} + C_2A_{22}C_2^{-1} \end{array} \tag{11}$$

The below LMI must be feasible:

$$\overline{A}_{11}^T\Gamma_1 + \Gamma_1\overline{A}_{11} < 0; \Gamma_1 = \Gamma_1^T > 0 \tag{12}$$

This property is key a for robustness. Many systems carry out it (power systems and electromechanical systems):

$$\overline{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}; \overline{B}_2 = C_2B_2 \in R^{p-m} \tag{13}$$

$$\overline{\Omega}(t, W, U) = \begin{bmatrix} \Psi_1(t, W) \\ C_1\Psi_1(t, W) + C_2\Psi_2(t, W, U) \end{bmatrix} \tag{14}$$

$$\Psi_2(t, W, U) = B_2\zeta_2(t, W, U) \tag{15}$$

where $\zeta_2(t, W, U)$ is a matched disturbance.

$$\parallel \zeta_2(t, W, U) \parallel = \gamma_1 \parallel U(t) \parallel + \alpha(t.Y) \tag{16}$$

Where the $rank(\overline{B}_2) = m$, $0 \leq \gamma_1 < 1$ is a positive constant, $\alpha(t.y)$ is a known nonnegative function.

The new system can be written as:

$$\begin{array}{l} \dot{X}_1(t) = \overline{A}_{11}X_1t + \overline{A}_{12}Y(t) + \Psi_1(t, W) \\ \dot{Y}(t) = \overline{A}_{21}X_1(t) + \overline{A}_{22}Y(t) + \overline{B}U(t) + C_1\Psi_1(t, W) \\ \quad +C_2B_2\zeta_2(t, Y, U) \end{array} \tag{17}$$

The sliding mode control problem can be established as:

" To design a sliding Mode Controller that only uses output feedback to stabilize the system described by (17) and provides it robustness".

The controller comprises a linear output feedback to regulate the system and overcome unmatched disturbances (it is possible), and a nonlinear controller to reject the matched disturbance.

III. Sliding model surface

The sliding mode surface chosen was:

$$\begin{array}{l} z(t) = ZY(t) = 0 \\ Z = [0\ Z_2] \end{array} \tag{18}$$

$Z_2 \in R^{m*m}$ nonsingular, Z must be chosen as $ZC_2B_2 = \Lambda \in R^{m*m}$ nonsingular diagonal matrix.

The sliding surface dynamic is:

$$\begin{array}{l} \dot{z}(t) = ZY(t) = Z\overline{A}_{21}X_1(t) + Z\overline{A}_{22}Y(t) \\ \quad +\Lambda U(t) + ZC_1\Psi_1(t, X) + \Lambda\zeta_2(t, X) \end{array} \tag{19}$$

IV. Controller Design

The proposed controller is:

$$\begin{array}{l} U(t) = U_l(t) + U_n(t) \\ U_l(t) = -KY(t) \\ U_n(t) = -\rho(t, Y)\Lambda^{-1}\frac{z(t)}{\|z(t)\|} \end{array} \tag{20}$$

Where $z(t) \neq 0$, $K \in R^{m*p}$ Feedback gain.

$$\rho(t, Y) = \delta_0 + \delta_1 \parallel Y(t) \parallel + \delta_2\alpha(t, Y) \tag{21}$$

$\delta_0 \geq \frac{\eta}{1-\gamma_1}$; η positive constant.
$\delta_1 \geq \frac{\|\Lambda Z A_{22} - \Lambda K\| + \Psi_{10}\|ZC_1\| + \gamma_1\|\Lambda\|\|K\|}{1-\gamma_1}$
$\delta_2 \geq \frac{\|\Lambda\|}{1-\gamma_1}$

V. Robustness Analysis

The nonlinear control component comes from the below sliding condition [6]:

$$\dot{V}(z) \leq -\eta \parallel z(t) \parallel \tag{22}$$

$V(z) = \frac{z^T(t)z(t)}{2}$ is a Lyapunov function. The K gain of linear control component satisfies the LMI:

$$(\overline{A} - \overline{B}K\overline{C})^TP + P(\overline{A} - \overline{B}K\overline{C}) + 2\varepsilon_0P < 0 \tag{23}$$

$P = P^T > 0$ positive define.

$$P = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix}; \ P_{11} = P_{11}^T > 0; \ P_{22} = P_{22}^T \tag{24}$$

$$\varepsilon_0 = \frac{\Psi_{10} \parallel T^{-1} \parallel \left(\parallel P_{11} \parallel + \parallel P_{22}C_1 \parallel \right)}{\lambda_{min}(P)} \tag{25}$$

If $m < p$, P_{22} must satisfy:

$$p_{22} = \begin{bmatrix} P_{2211} & P_{2212} \\ P_{2221} & P_{2222} \end{bmatrix} \tag{26}$$

With $P_{2212} = -P_{2211}h_{12} = P_{2221}^T \in R^{(p-m)*m}$, $P_{2222} > h_{12}^TP_{2211}h_{12}$.

$$\Gamma_{22} = P_{2222} - h_{12}^TP_{2211}h_{12} \tag{27}$$

It is necesary that: $\frac{\lambda_{min}\left(\Gamma_{22}\right)}{\lambda_{max}\left(\Gamma_{22}\right)} > \gamma_1$.

From (23):

$$\left(\overline{A}_{22} - \overline{B}_2 K\right)^T P_{22} + P_{22}\left(\overline{A}_{22} - \overline{B}_2 K\right) < 0 \qquad (28)$$
$$\overline{A}_{22Lc} = \overline{A}_{22} - \overline{B}_2 K$$

$\overline{A}_{22Lc}$ Can be chosen with all its eigenvalues with negative real part.

It must satisfy $rank\left(\left[\overline{A}_{22} - \overline{A}_{22Lc} \;\; \overline{B}_2\right]\right) = m$.

VI. EXAMPLES

A. Example 1

Consider the system:

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -4x_1(t) - 3x_2(t) + x_3(t) \\
\dot{x}_3(t) &= 2x_3(t) + 3cos(x_1(t))u_1(t) + x_3(t)sin(x_4(t)) \\
\dot{x}_4(t) &= x_1(t) - 3x_2(t) + 3x_4(t) - 2.5cos(x_2(t))u_2(t) \\
&\quad + x_4(t)cos(x_1(t))
\end{aligned} \qquad (29)$$

$$\begin{aligned} y_1(t) &= x_3(t) \\ y_2(t) &= x_4(t) \end{aligned} \qquad (30)$$

This example corresponds to a system nonlinears with matched uncertainties. There is the restriction:

$$-\tfrac{\pi}{3} \leq x_1 \leq \tfrac{\pi}{3}; \; -\tfrac{\pi}{3} \leq x_2 \leq \tfrac{\pi}{3}$$

Identifying the basic structure described by (1) the values are $n = 4$; $m = 2$; $p = 2$ and:

$$\begin{aligned} X &= [x_1 \; x_2 \; x_3 \; x_4]^T \in R^4 \\ Y &= [x_3 \; x_4]^T \in R^2 \\ U &= [u_1 \; u_2]^T \in R^2 \end{aligned} \qquad (31)$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & -3 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & -3 & 0 & 3 \end{bmatrix} \in R^{4*4} \qquad (32)$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 3 & 0 \\ 0 & -2.5 \end{bmatrix} \in R^{4*2} \qquad (33)$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in R^{2*4} \qquad (34)$$

The rank condition is fullfiled: $rank(B) = 2$; $rank(C) = 2$.

$$\begin{aligned}
A_{11} &= \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix} \in R^{2*2} \\
A_{12} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in R^{2*2} \\
A_{21} &= \begin{bmatrix} 0 & 0 \\ 1 & -3 \end{bmatrix} \in R^{2*2} \\
A_{22} &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \in R^{2*2}
\end{aligned} \qquad (35)$$

$$B_2 = \begin{bmatrix} 3 & 0 \\ -2 & -5 \end{bmatrix} \in R^{2*2} \qquad (36)$$

$$\begin{aligned}
C_1 &= \begin{bmatrix} 0 & 0 \\ 0 & -0 \end{bmatrix} \in R^{2*2} \\
C_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in R^{2*2}
\end{aligned} \qquad (37)$$

In accordance with above: $T = I_{4*4}$; $\overline{A} = A$; $\overline{B} = B$; $\overline{C} = C$

The LMI (12) is satisfied:

$$\begin{aligned}
\Psi_1(t, X) &= 0 \\
\Psi_2(t, x, U) &= \begin{bmatrix} (cosx_1 - 1)u_1(t) + x_3 sinx_4/3 \\ (cosx_2 - 1)u_2(t) - x_4 cosx_1/2.5 \end{bmatrix}
\end{aligned} \qquad (38)$$

$$\begin{aligned} \Psi_2(t, X, U) &= \delta(X) + \zeta(X, U) \\ \zeta(X, U) &= \zeta_2(X)U(t) \end{aligned}$$

$$\begin{aligned}
\delta(X) &= \begin{bmatrix} x_3 sin(x_4)/3 \\ x_4 sin(x_1)/2.5 \end{bmatrix} \\
\zeta_2(X) &= \begin{bmatrix} cos(x_1) - 1 & 0 \\ 0 & cos(x_2) - 1 \end{bmatrix}
\end{aligned} \qquad (39)$$

$$\begin{aligned}
&\parallel \delta(X) \parallel \leq \alpha(Y) = |x_3|/3 + |x_4|/2.5 \\
&\parallel \zeta(X, U) \parallel \leq \zeta_{20} \parallel U(t) \parallel \\
&\zeta_{20} = max\{|1 - cos(x_1)|, |1 - cos(x_2)|\}
\end{aligned} \qquad (40)$$

With the restriction on x_1, x_2:

$$\begin{aligned} \gamma_1 &= 0,5 \\ \zeta_{20} &\leq 0.5 \end{aligned} \qquad (41)$$

The parameters are:

$$\begin{aligned}
K &= \begin{bmatrix} 1.33 & 0 \\ 0 & -3.2 \end{bmatrix} \\
\rho(Y) &= 3 + 8 \parallel Y(t) \parallel + 3\alpha(Y)
\end{aligned} \qquad (42)$$

The system was simulated in Simulink/Matlab and the results are shown below in the figures 1 to 6.

B. Example 2

Consider the system:

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -x_1(t) - 2x_2(t) + x_3(t)\left(1 + 0.5sin(10x_1)\right) \\
\dot{x}_3(t) &= -x_1(t) + 3x_2(t)\left(1 + x_3(t)sin(x_2)\right) + \\
&\quad \left(3 + 1.2sin(x_2 x_3)\right)u(t)
\end{aligned} \qquad (43)$$

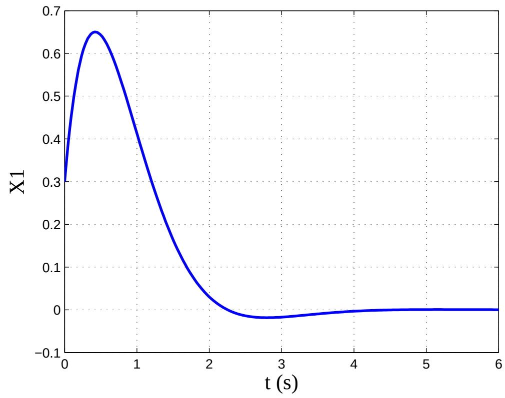

Fig. 1. State X1 as a function of time

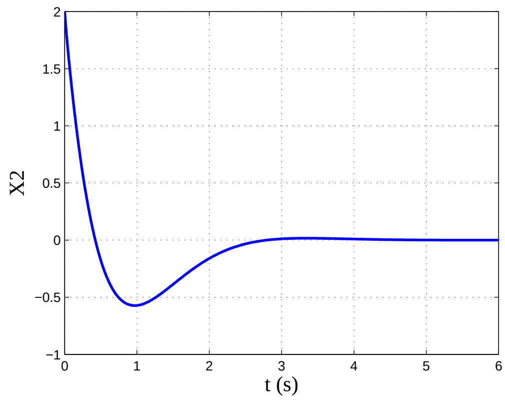

Fig. 2. State X2 as a function of time

$$\begin{aligned} y_1(t) &= x_2(t) \\ y_2(t) &= x_3(t) \end{aligned} \tag{44}$$

Here $n = 3$, $m = 1$, $p = 2$ and:

$$\begin{aligned} X &= [x_1\ x_2\ x_3\ ]^T \in R^3 \\ Y &= [x_2\ x_3]^T \in R^2 \\ U &= [u] \in R \end{aligned} \tag{45}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 1 \\ -1 & 3 & 0 \end{bmatrix} \in R^{3*3} \tag{46}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \in R^3 \tag{47}$$

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in R^{2*3} \tag{48}$$

The rank condition is fulfilled: $rank(B) = 1$; $rank(C) = 2$.

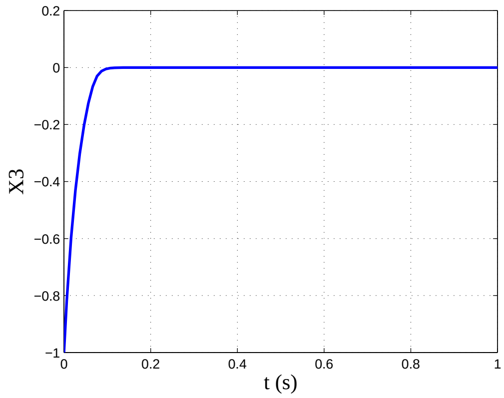

Fig. 3. State X3 as a function of time

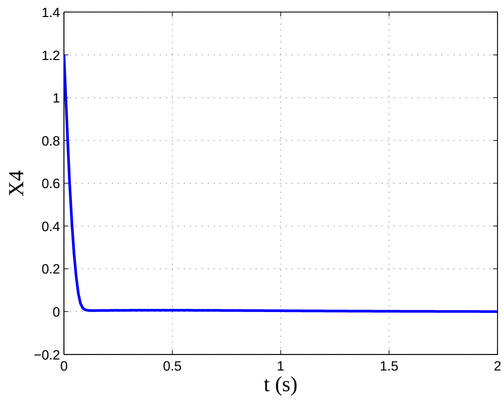

Fig. 4. State X4 as a function of time

$$\begin{aligned} A_{11} &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \in R^{2*2}; \\ A_{12} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in R^{2*1} \\ A_{21} &= \begin{bmatrix} -1 & 3 \end{bmatrix} \in R^{1*2}; \\ A_{22} &= 0 \in R \end{aligned} \tag{49}$$

$$B_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \in R^{2*1} \tag{50}$$

$$\begin{aligned} C_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in R^{2*1}; \\ C_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in R^{2*2} \end{aligned} \tag{51}$$

$C_{211} = 1$; $C_{212} = C_{221} = 0$; $C_{222} = 1$.

In accordance with above:

$T = I_{3*3}$; $\overline{A} = A$; $\overline{B} = B$; $\overline{C} = C$

The LMI (12) is satisfied.

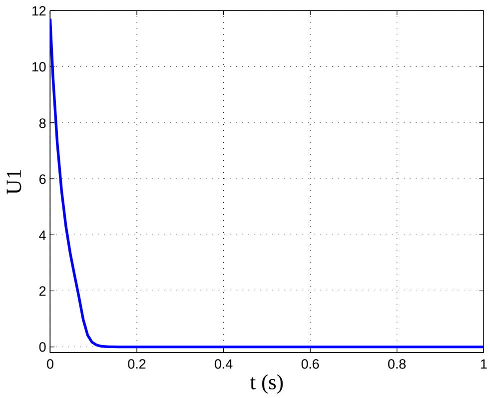

Fig. 5. Control signal U1 as a function of time

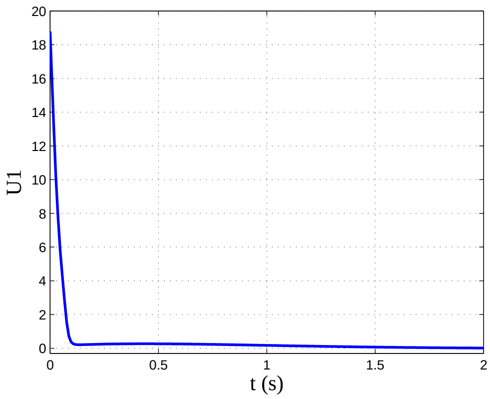

Fig. 6. Control signal U2 as a function of time

$$
\begin{aligned}
&\Psi_1(t,X)=0\\
&\Psi_2(t,X,u)=\left[\begin{array}{c} 0.5x_3 sin(10x_1)\\ 3x_2x_3 sinx_2+1.2sin(x_2x_3)u(t)\end{array}\right]
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
&\Psi_2(t,X,u)=\delta(X)+\zeta(X,u)\\
&\zeta(X,u)=\zeta_2(X)u(t)
\end{aligned}
\tag{53}
$$

$$
\delta(X)=\left[\begin{array}{c} 0.17x_3 sin(10x_1)\\ x_2x_3 sinx_2\end{array}\right]
\tag{54}
$$

$$
\begin{aligned}
&\parallel \delta(X)\parallel \leq 0.17|x_3|+|x_2|*|x_3|\\
&\zeta_2(X)=\left[\begin{array}{c} 0\\ 0.4sin(x_2x_3)\end{array}\right]
\end{aligned}
$$

With $\alpha(Y)=0.17|x_3|+|x_2|*|x_3|$, $\gamma_1=0.5$, $\zeta_{20}\leq 0.5$.

The parameters are:

$$
\begin{aligned}
&K=\left[\begin{array}{cc} 1.33 & 1\end{array}\right]\\
&\rho(Y)=2+15\parallel Y(t)\parallel+5\alpha(Y)
\end{aligned}
\tag{55}
$$

The system was simulated in Simulink/Matlab and the results are shown below in the figures 7 to 10.

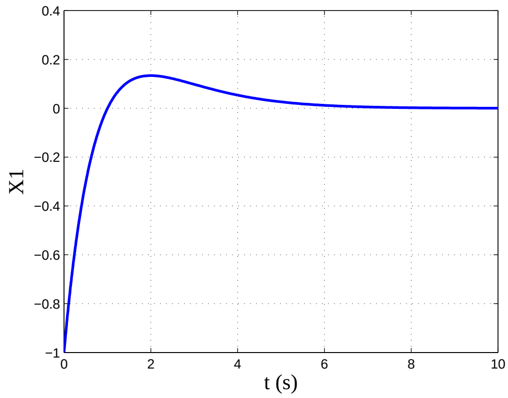

Fig. 7. State X1 as a function of time

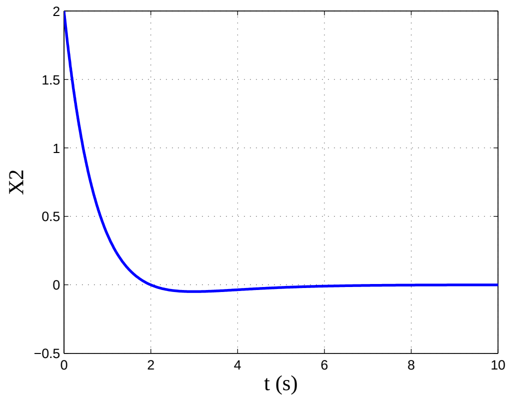

Fig. 8. State X2 as a function of time

VII. Conclusions

In this paper a sliding mode controller with output feedback only was shown and the conditions for overall robustness of the system were given.

It is important to annotate that only some systems fulfill the required conditions introduced here. But several practical systems can fit in this category.

Systems with the propierty $m=p$ (square systems) are easier to control than systems with $m<p$.

If the conditions given here are not satisfied, local stability can be guaranteed and the controller wi1l stabilize the system into a part of state space.

VIII. Acknowledgments

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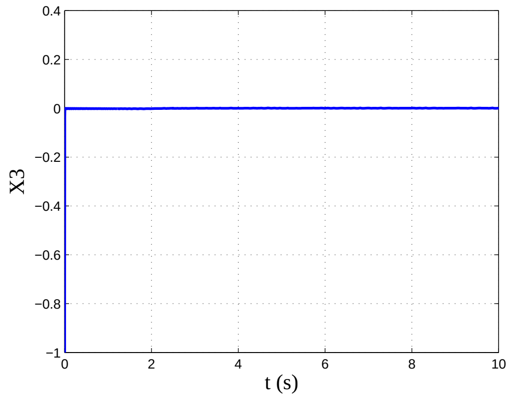

Fig. 9. State X3 as a function of time

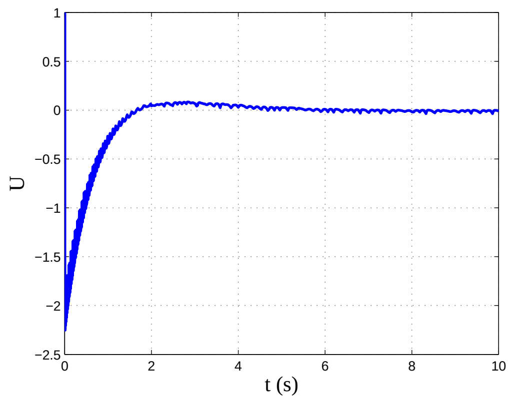

Fig. 10. Control signal U as a function of time

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