Synthesis of H $_\infty$ PID controllers

P. Burbano[1], Member, IEEE, V. Proaño[2]

[1]Dept. of Automation and Industrial Electronics, Escuela Politécnica Nacional, Isabel La Católica S/N Quito, Ecuador, email: pburbano@mailfie.epn.edu.ec[1]

[2] Dept. of Automation and Control, Escuela Politécnica del Ejército, Quito, Ecuador, email: vgproano@espe.edu.ec

Abstract – **This paper considers the problem of synthesizing proportional-integral-derivative (PID) controllers for which the closed-loop system is internally stable and the H $_\infty$ norm of a related transfer function is less than a prescribed level for a given single-input single output plant. In our approach, in this paper, this problem is formulated by virtue of the bounded real lemma, as a feasibility problem involving a** bilinear matrix inequality **(BMI) and solved by means of a sequence of easier auxiliary quadratic optimization problems. The optimal controller is determined through a line search on the non negative real axis.**

I. Introduction

Today, many controller design methods are based on minimization of a criterion. Controllers designed in this way have been shown to have good general robustness properties. In practice, any model is an inaccurate representation of the true process. Robust control addresses this plant/model mismatch by defining a set of plants of which the true process is an element. This set is defined by an uncertainty description. Controllers are designed to be robust to the uncertainty, that is, to achieve a desired level of performance for any plant in the set.

The PID controller has several important functions: it provides feedback; it has the ability to eliminate state offsets through integral action; it can anticipate the future through derivative action. PID controllers are sufficient for many control problems, particularly when process dynamics are benign and the performance requirements are modest. PID controllers are found in large numbers in all industries. Its structural simplicity and sufficient ability of solving many control problems have greatly contributed to this wide acceptance. Many formulas for optimal PID controller designs can be found in the literature, and [1] provides an excellent review. PID control is an important ingredient of a distributed control system. PID control is often combined with logic, sequential machines, selectors, and simple function blocks to build the complicated automation systems used for energy production, transportation, and manufacturing. The PID controller can thus be said to be the "bread and butter" of control engineering. It is an important component in every control engineer's toolbox. PID controllers have survived many changes in technology ranging from pneumatics to microprocessors via electronic tubes, transistors, integrated circuits. The microprocessor has had a dramatic influence on the PID controller. Practically all PID controllers made today are based on microprocessors. The emergence of the fieldbus is another important development. The PID controller is an important ingredient of the fieldbus concept.

A large cadre of instrument and process engineers is familiar with PID control. There is a well-established practice of installing, tuning and using the controllers. In spite of this there are substantial potentials for improving PID control. Evidence for this can be found in the control rooms of any industry. Many controllers are put in manual mode, and among those that are in the automatic mode, derivative action is frequently switched off for the simple reason that is difficult to tune properly. Knowledge and understanding are the key elements for improving performance of the control loop. Specific process knowledge is required as well as knowledge about PID control.

H $_\infty$ refers to the space of stable and proper transfer functions. We generally desire that the closed loop transfer functions be proper and stable, we say $P(s)$ is in H $_\infty$. The basic object of interest in H $_\infty$ control is a transfer function [2]. In fact, we will be optimizing over the space of transfer functions. Optimization presupposes a cost (or objective) function, because we want to compare different transfer functions and choose the best one in that space. In H $_\infty$ control, we compare transfer function according to their ∞-norm. The ∞-norm of a transfer function is defined by

$$\|P\|_\infty = \sup_w |P(w)| .$$

This is easy to compute graphically, it is simply the peak in the Bode magnitude plot of the transfer function. For example the multiplicative stability margin (MSM) or the smallest destabilizing uncertainty can be written as:

$$MSM = \frac{1}{\|T\|_\infty}$$

In H $_\infty$ control, the objective is to minimize the ∞-norm of T. Note that this will increase the robust stability margin of the system.

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II. PRELIMINARIES

It is important to review some relevant aspects of Linear Matrix Inequalities (LMI) and robustness criterions previous to formulation problem.

A. Linear Matrix Inequalities

A Linear Matrix Inequality (LMI) has the form:

$$F(x) \overset{\Delta}{=} F_0 + x_1 F_1 + \cdots + x_m F_m > 0 \quad (1)$$

Where $x \in R^m$; $x = [x_1 \ldots x_m]^T$ is a vector of real numbers. $F_0, \ldots, F_m$ are real symmetric matrices, i.e., $F_i = F_i^T \in R^{nxn}, i = 0, \ldots, m$. The inequality >0 means 'positive definite', i.e., $u^T F(x) u > 0$ for all $u \in R^n$. Equivalently, the smallest eigenvalue of $F(x)$ is positive.

Many optimization problems in control design, identification and signal processing can be formulated using linear matrix inequalities. Clearly, it only makes sense to cast these problems in terms of LMI's if these inequalities can be solved efficiently and in a reliable way. Since the linear matrix inequality F(x)>0 defines a convex constraint on the variable x, optimization problems involving the minimization of a performance function: $f: C \rightarrow \Re$ with C: ={x/F(x)>0} belongs to the class of convex optimization problems. It may be apparent that the full power of convex optimization theory can be employed if the performance function f is known to be convex.

One of the generic problems related to the study of linear matrix inequalities is the feasibility problem. It test whether or not there exist solutions x of F(x)>0. The LMI is called non-feasible if no solutions exist.

A bilinear matrix inequality (BMI) is of the form

$$F(x,y) = F_0 + \sum_{i=1}^{m} x_i F_i + \sum_{j=1}^{k} y_j G_j + \sum_{i=1}^{m} \sum_{j=1}^{k} x_i y_j H_{ij} \geq 0 \quad (2)$$

Where G_j and H_{ij} are symmetric matrices of the same dimension as F_i, and $y \in \Re^k$.

The bilinear terms make the set $\{x, y | F(x,y) \geq 0\}$ nonconvex and no off-the-shelf software exist for solving optimization problems with BMI constraints. It is straightforward to prove that BMI optimization problems are NP-hard, which implies that is highly unlikely that there exists a polynomial-time algorithm for solving these problems.

Here we use the Correa-Sales numerical algorithm in order to solve the BMI problem.

Relying upon a sequence of auxiliary quadratic problems, the algorithm finds, if any, a feasible point on the boundary of the feasible BMI constraint set. Roughly speaking, a problem which has an infinite number of constraints, will be solved via a sequence of problems, each one of these has one single weighted constraint.

B. Robustness

Robustness requires that stability must be maintained despite model uncertainties: structured and unstructured.

Any process model is only an approximation of the true process. In robust control, the true plant P is covered by a set of plants $\widetilde{P}$ which is represented by the nominal model P and a set of norm bounded disturbance Δ.

For a multiplicative uncertainty Δ_m

$$\Delta_m = \frac{\widetilde{P} - P}{P} \quad (3)$$

Figure 1 shows a feedback control loop in which P represents the nominal model of the plant, subject to model uncertainties and disturbances.

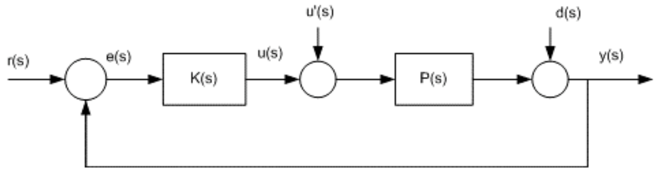

Fig.1 Feedback control system

A robust control system provides stable, consistent performance as specified by the designer in spite of wide variations of plant uncertainties and disturbances. It also provides highly robust response to command inputs and a steady-state tracking error equal to zero [4], [5].

Robust control emerges when the machine intelligence requirements are in between highly and moderate because of model uncertainties and disturbances are in between highly and moderate.

The ultimate requirement to the compensator is that it works 'well' for the real system. These requirements can be subdivided into the following four categories:

Nominal stability: the compensator must ensure internal stability in the controlled system, provided the model is correct.

Nominal performance: the compensator must minimize the error e. The H_∞ optimal control minimizes the H_∞ norm of SW, where W is a frequency dependent weight and S is the sensitivity function.

Robust stability: for all models in $\widetilde{P}$ the compensator must ensure that the error is within a specified bound. The compensator K provides robust stability if and only if $\|WT\|_\infty < 1$ where W is a fixed stable transfer function, the weight and T is the complementary sensitivity function

$$T = 1 - S = \frac{KP}{1 + KP} \quad (4)$$

Robust performance: for all models in $\widetilde{P}$ the compensator must ensure that the error is within a specified bound.

III. PROBLEM FORMULATION

The main objective of this paper is to design a robust PID controller using the LMI methodology in a state space framework.

A. Problem Setup

Because of the PID controller doesn't have a state space realization, Equation 5 is used to obtain Equations 6

$$\frac{1}{s}K(s)=\frac{K_d s^2+K_p s+K_i}{s^2} \quad (5)$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}w$$

$$u=\begin{bmatrix}K_i & K_p\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+K_d w \quad (6)$$

The sP(s) realization is shown in Equation 7

$$sP(s)=P_p(s)\sim(A_p,B_p,C_p,D_p) \quad (7)$$

$$D_p=CB;A_P=A;C_p=C;B_p=AB$$

The closed loop description of the system is show in Equation 8.

$$\begin{bmatrix}\dot{x}_p\\ \dot{x}_c\end{bmatrix}=A_{cl}\begin{bmatrix}x_p\\x_c\end{bmatrix}+B_{cl}r \quad (8)$$

$$y=C_{cl}\begin{bmatrix}x_p\\x_c\end{bmatrix}$$

Where:

$$A_{cl}=\begin{bmatrix}A_p-B_pD_cC_p & B_pC_c\\ -B_cC_p & A_c\end{bmatrix};B_{cl}=\begin{bmatrix}B_pD_c\\B_c\end{bmatrix};C_{cl}=\begin{bmatrix}C_p & 0\end{bmatrix}$$

Applying the Bounded Real Lemma [7], we convert the frequency formulation to a BMI

$\|T_{rz}(C_c,D_c)\|_\infty\le\gamma$ iff:

$$\begin{bmatrix}A_{cl}^TP+PA_{cl} & C_{cl}^T & PB_{cl}\\ C_{cl} & -I & 0\\ B_{cl}^TP & 0 & -\gamma^2 I\end{bmatrix}\le 0 \text{ y } P\ge 0 \quad (9)$$

B. Synthesis of the controller

In order to discriminate the unknowns from the constant terms, the last are included in the H and L matrices.

$$H_{00}+H_2PH_3+H_2^TPH_3^T+H_4D_c^TH_1^TPH_3+$$
$$H_3^TPH_1D_cH_4^T+H_5C_c^TL_2^TPH_3+\ldots$$
$$+H_3^TPL_2C_cH_5^T\le 0$$
$$-P\le 0$$

Where P is the Lyapunov matrix, Dc and Cc are the variable matrices with unknown parameters Kp, Ki, Kd which should be searched.

The H and L matrices are constant terms obtained from the BMI of Equation 9. In order to determine a solution to this problem, the following one is introduced

$$\min_{P,Dc,Cc}\|P-P_q\|_2^2+\|C_c-C_q\|_2^2+\|D_c-D_q\|_2^2$$

Subject to $B(P,D_c,C_c)\le 0$

Note that this problem is characterized by an auxiliary quadratic cost and the same BMI constraint. Solutions will be pursued on the basis of a sequence of auxiliary quadratic problems subject to inequality bilinear constraints, defined as follows

$$\min_{P,Dc,Cc}\|P-P_q\|_2^2+\|C_c-C_q\|_2^2+\|D_c-D_q\|_2^2$$

Subject to $\langle W,WB(P,D_c,C_c)\rangle\le 0;$

Note that the infinite number of constraints in the original problem is replaced by a single one, since the element W is fixed.

Roughly speaking, the original problem witch has an infinite number of constraints, will be solved via a sequence of problems each one of these has one single weighted constraint.

Using the Lagrangian

$$g\langle\lambda\rangle=\inf_{P,D_c,C_c} L(P,D_c,C_c,\lambda)$$

$$L(P,D_c,C_c,\lambda)=\|P-P_q\|_2^2+\|C_c-C_q\|_2^2+\ldots$$
$$\|D_c-D_q\|_2^2+$$
$$\lambda\langle W_1(H_{00}+H_2PH_3+H_3^TPH_2^T+H_4D_c^TH_1^TPH_3)W_1\rangle+$$
$$+\lambda\langle W_1(H_3^TPH_1D_cH_4^T+H_5C_c^TL_2^TPH_3+H_3^TPL_2C_cH_5^T),W_1\rangle$$
$$+\lambda\langle W_2,-PW_2\rangle$$

The solution of the problem is obtained essentially by finding the root of a scalar nonlinear equation of one variable.

In order to solve the BMI problem we use the following conceptual numerical algorithm [3].

1. Fix $\varepsilon>0$, set $k:=1$; and initialize $P_q^{\langle k\rangle},D_q^{\langle k\rangle},C_q^{\langle k\rangle}$.
2. Let $W^{\langle k\rangle}=vv^T$ where v is the eigenvector corresponding to the largest eigenvalue of $B(P_q^{\langle k\rangle},D_q^{\langle k\rangle},C_q^{\langle k\rangle})$

3. Solve $\min\limits_{P,Dc,Cc} \left\|P - P_q\right\|_2^2 + \left\|C_c - C_q\right\|_2^2 + \left\|D_c - D_q\right\|_2^2$ subject to $\left\langle W^{\langle k\rangle}, W^{\langle k\rangle} B\left(P_q^{\langle k\rangle}, D_q^{\langle k\rangle}, C_q^{\langle k\rangle}\right)\right\rangle \le 0$; Let

$P_*^{\langle k\rangle}, C_*^{\langle k\rangle}, D_*^{\langle k\rangle}$ be its solution.

4. Compute $\overline{\lambda}\left(B\left(P_*^{\langle k\rangle}, D_*^{\langle k\rangle}, C_*^{\langle k\rangle}\right)\right)$ where $\overline{\lambda}$ Denote the greatest eigenvalue of the real symmetric matrix B. If it is less than or equal to ε then stop. Else update $P_q^{\langle k\rangle} = P_*^{\langle k\rangle}, D_q^{\langle k\rangle} = D_*^{\langle k\rangle}, C_q^{\langle k\rangle} = C_*^{\langle k\rangle}$, set $k := k+1$, and repeat from step 2.

IV. SIMULATION RESULTS

The numerical algorithm to solve the BMI problem using the Correa-Sales methodology was implemented in a MATLAB program. This program tries to minimize the $\left\|T_{rz}(C_c, D_c)\right\|_\infty$ which, in some cases, represents the complementary sensitivity and in other ones the sensitivity.

In this paper we applied the methodology on two case studies: a typical unstable system, a type 2 system and the rotational velocity of a satellite [6].

In addition, this paper solves two problems: robust stability and nominal performance.

A. Type 2 system

The plant transfer function is $P(s) = \dfrac{1}{s^2}$. Here the objective is to minimize the complementary sensitivity to achieve robust margin stability.

The strategy starts with a large γ handy manipulated and then after a few iterations we obtain the following results for $\gamma = 1.2$.

Ki=0.08, Kp=1.21, Kd=2.42.

Figure 2 shows the Bode diagram for the closed loop complementary sensitivity. The peak represents $\left\|T_{rz}\right\|_\infty$.

Another numerical example of this case study is shown in Figure 3 in which the sensitivity is minimized and represents a nominal performance objective considering the rms value of any input The initial values of the controller were set randomly. These examples show the goodness of the algorithm, because the obtained response is in the worst case.

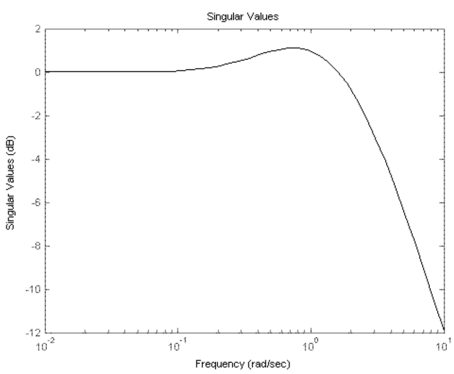

Fig.2 Bode diagram of complementary sensitivity

The result considers an input step signal. The minimization process is aborted up to $\gamma = 1.9$ with (for a Mp=20%):

Ki=0.08, Kp=1.29, Kd=2.57.

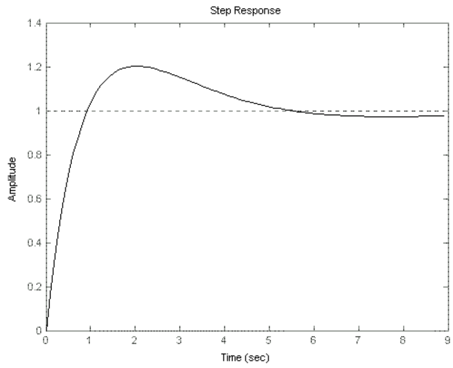

Fig.3 Step response

Figures 4,5 shows the frequency response when a multiplicative uncertainty is added. Figure 4 plots the inverse return difference before (dash) and after the optimization (solid); and the multiplicative uncertainty (dot). Figure 5 plots the same response but with zoom.

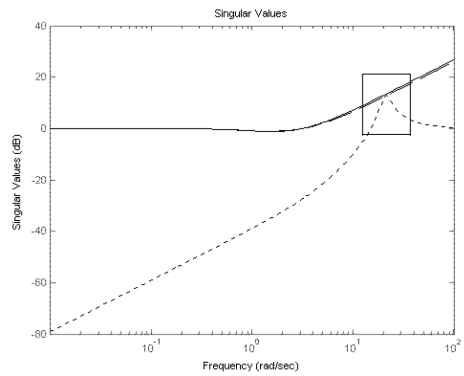

Fig.4 Plots oft the inverse return difference before (dash) and after optimization (solid) and the multiplicative uncertainty (dot).

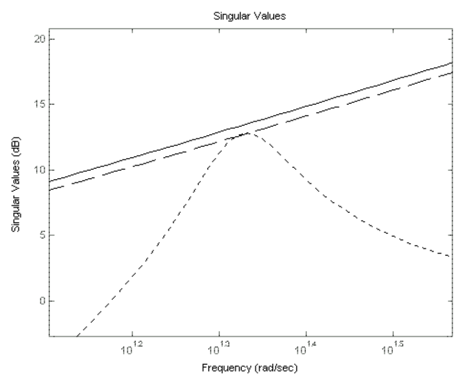

Fig. 5 Zoom of figure 4 verifying robust stability.

After optimization the multiplicative uncertainty doesn't cut the inverse return difference plot which means that it guarantees robust stability which is shown in time plots of figures 6,7..

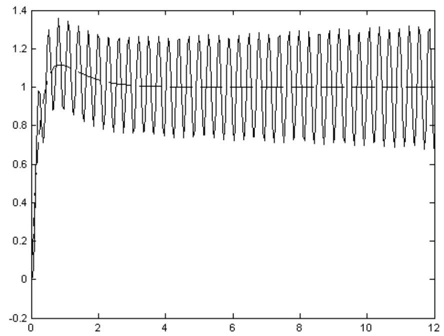

Fig. 6 Step response before optimization

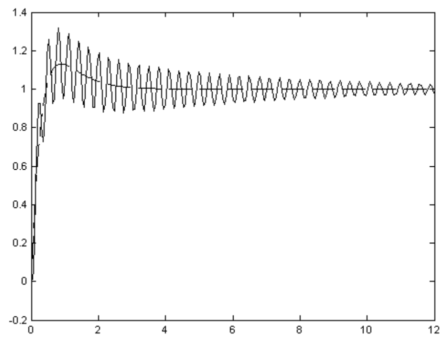

Fig. 7 Step response after optimization

Another interesting problem is when the uncertainty is a delay type because of it's known that is a difficult control problem of non minimum phase system.

Figure 10 shows that the original time response retains its dynamic characteristic (it is not degraded), but when delay uncertainty is added, as the above problem, the optimal design guarantees robust stability. In this case it's possible to improve the time optimal response with a faster transient, changing the original design with a lower bandwidth and trying another optimization problem, which it is not important by now because we are interested in robust stability. The last frequency response (solid) guarantees robust stability.

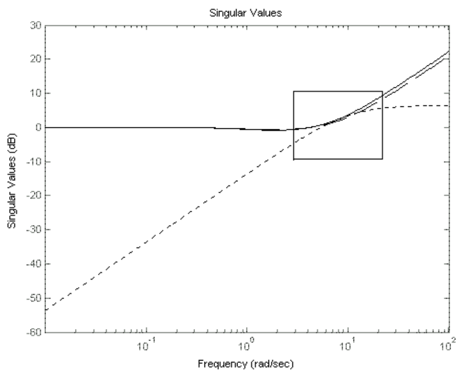

Fig.8 Plots oft the inverse return difference before (dash) and after optimization (solid) and the multiplicative uncertainty (dot).

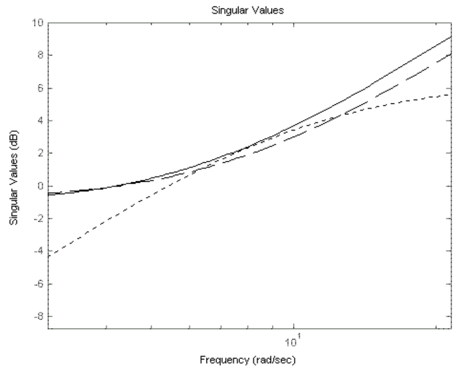

Fig. 9 Zoom of figure 8 verifying robust stability

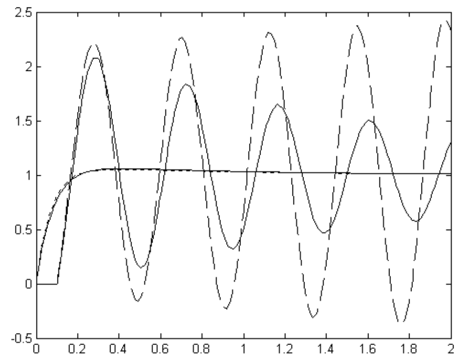

Fig. 10 Step responses with and without optimization.

B. *Satellite model*

The rotational velocity with respect to an incremental change in beam length adjustment is represented by [4]:

$$P(s) = \frac{2.5(s+2)}{(s+5)(s+1)^2}$$

The frequency response is used in order to minimize the complementary sensitivity H_∞ norm, $\|T\|_\infty$. This fact means that we are maximizing the multiplicative uncertainty the system can support without get unstability. Figure 11 shows the Bode diagram with $\gamma = 10$. The PID parameters are:

Ki=0.77, Kp=0.91, Kd=0.99

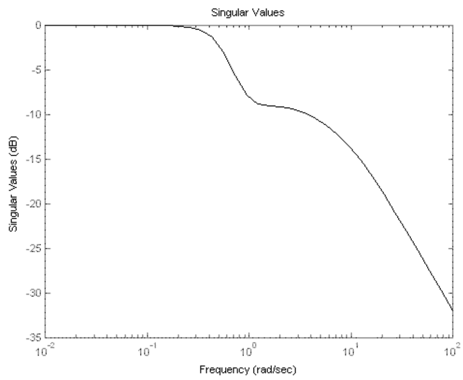

Fig.11. Complementary sensitivity

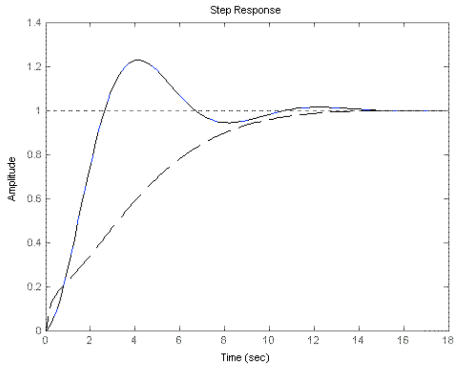

Fig.12. Original (solid) and optimized (dash) step responses.

In this model we solve the nominal performance optimization using an initial design with high overshoot. The objective is to minimize the sensitivity H_∞ norm, $\|S\|_\infty$, which is known as a good robustness measure.

The optimization algorithm keeps the settling time but improves the response. As it is known the improvement of the properties of a control system in one respect will bring deterioration in another but in this case the response is not degraded. This fact is shown in figure 12.

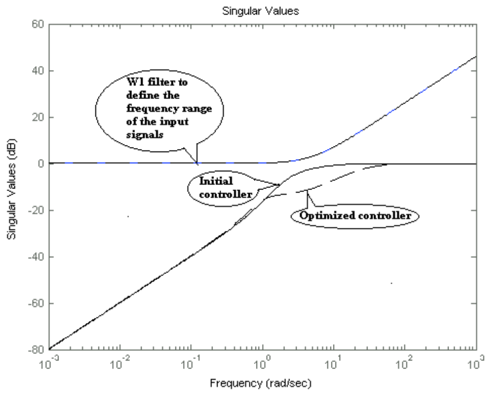

Fig.13. Original (solid) and optimized (dash) responses

In figure 13 we minimize the norm $\|W_1S\|_\infty$. W_1 is a first order filter that means the frequency range of disturbances the system must reject. We see the optimal controller improves the attenuation of medium frequencies perturbations.

V. CONCLUSIONS

LMI approach is an efficient methodology to design robust controllers. A procedure for synthesis of PID H_∞ controllers has been presented. The problem was rewritten into bilinear matrix inequality (BMI) form resulting in the unknowns P,D,C, where P is the Lyapunov matrix and D,C have the PID parameters Ki,Kp,Kd. Then the problem is solved using the Correa-Sales algorithm, relying upon a sequence of auxiliary quadratic problems. It is possible to make several and different simulation results based on robust stability and nominal performance requirements, which are depending on the user needs. Here we present simulation results of the BMI formulation and numerical results applying the Correa-Sales algorithm.

VI. ACKNOWLEGEMENTS

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VI. REFERENCES

[1] K. J. Astrom, T. Hagglund, PID Controllers, 2nd Edition, USA, 1995

[2] M. Hassul, B. Shaian, Control System Design Using Matlab, USA. 1995.

[3] D. Mideiros .Sales, G. O. Correa, A New Approach to the H∞ with Static Output Feedback Synthesis Problem, Brazil, 2002..

[4] R. C. Dorf, R.H. Bishop, Modern Control Systems, Seventh Edition, Addisson Wesley, USA, 1997..

[5] S. Toffner-Clause, P. Andersen, J. Stoustrup, Robust Control, USA, 2001.

[6] V. Proaño, Diseño de Controladores PID con Especificaciones de Desempeño H Infinito, MSc. Thesis, Ecuador, 2005

[7] C. Scherer, S. Wieland, Lecture Notes DISC course on Linear Matrix Inequalities in Control. USA, 1999