A NEW FORMULATION OF RELATIVISTIC DYNAMICS

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ABSTRACT

This is a continuation of a previous paper. A brief revision of Einstein's General Relativity is carried out and some of its inconsistencies are brought to light. The concept of the relativistic (invariant) force and its components is also considered. Then, the equations of the general motion in the relativistic euclidean 4-space (E-4D) is derived and applied to several types of motion. In the example relative to the rectilinear, inverse-square law motion, the coulombian attraction between two charged particles is interpreted from the relativistic standpoint. The example of central-force motion, worked out in some length, gives rise to several interesting questions, some of which are illustrated on an electric circuit model.

I. INTRODUCTION.

Albert Einstein fundamented his General Theory of Relativity on the Equivalence Principle in Minkowski space, according to which, "*a non inertial reference system is equivalent to a gravitational field* ". Minkowski space is characterized, in its turn, by a peculiar meaning given to the "distance" (interval) *s*, which, in the differential form and with reference to an inertial system, can be defined by the equation:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \qquad (1.1)$$

Unfortunately, both concepts, the Equivalence Principle and Minkowski space, generate a number of difficulties. Here are some of them.

A. With reference to the Equivalence Principle.

A1.- *Asymptotic behaviour:* A "true" gravitational field vanishes at the infinity, whereas the one equivalent to a reference frame associated, for example, with a freely falling body indefinitely increases or, at least, is kept constant, this latter being the case of a freely orbiting satelite.

A2.- A *gravitational field* equivalent to a reference system disappears and the system becomes inertial through a suitable coordinate transformation. On the other hand, a "true" gravitational field can be "inertialized" only in a small volume, where the field can be assumed uniform. Besides, even in this case, the description of the outer world will become distorted as a fair campus watched from a merry-go-round.

A3.- When dealing with a *general non inertial system*, one does not know a sufficient number of parameters for its complete description. In fact, to describe a non inertial system, a more general space than that of Minkowski must be used - for example Riemann space, characterized by the (infinitessimal) interval squared:

$$ds^2 = g_{ij}\, dx^i\, dx^j \qquad (1.2)$$

where the "dumb" indices *i*, *j*, run from 0 to 3. The fundamental metric tensor g_{ij} is symmetric, of rank 4, thus being defined by 10 independent components which cannot be determined by considering only the reference frame orientation in a 4-space.

A4.- The last (but not the least important) drawback of the *Equivalence Principle* is to reduce the physical laws to purely geometrical concepts. As it has been already pointed out in [1], even admitting that an abstract, static, Einstein-Minkowski space can exist, it is certainly not our space, i.e. a suitable environment for the dynamic universe we live in. In particular, there is no objective reason whatsoever for identifying the components g_{ij} of the metric tensor with the gravitational potentials.

B. With reference to Minkowski space.-

Einstein attempted to unify the formal description of the euclidean and Minkowski spaces by introducing the imaginary coordinate $x^4 = i\,ct$ $(i = \sqrt{-1})$. Alternatively, the unwieldiness of the complex arithmetics (and geometry) can be alleviated by defining the Galilean, metric, pseudo-orthonormal matrix

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (1.3)$$

By this artifice, the application of the *Least Action Principle* requires to compute the extremum of the integral

$$S = -mc\int ds \qquad (1.4)$$

with *ds* given in (1.2). The computation is carried out by solving the equation

$$\delta S = 0 \qquad (1.5)$$

and brings along the equation of the

geodesics:

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 \qquad (1.6)$$

where Γ^i_{jk} are the well known Christoffel symbols of 2^{nd} kind (see APPENDIX A).

Unfortunately again, equation (1.6) proves to be unbounded for a system with a speed comparable to that of light because, when ds approaches zero, all terms in (1.6) become infinite. This difficulty forced Einstein to look for an alternative solution, such as the one given by the *Riemann curvature tensor*, subject to the condition that the resulting equations have the attributes of the Poisson equation; that is to say, use at most second derivatives of the "gravitational potentials" g_{ij}. After a lot of computations, Einstein arrived at his famous "field Equation":

$$R_{ij} = \frac{1}{2} g_{ij} R = -\varkappa T_{ij} \qquad (1.7)$$

where

$$R_{ij} = -\frac{\partial \Gamma^k_{ij}}{\partial x^k} + \Gamma^k_{il} \Gamma^l_{jk} + \frac{\partial \Gamma^k_{ik}}{\partial x^j} - \Gamma^k_{ij} \Gamma^l_{kl} \qquad (1.8)$$

is the symmetric, (2^{nd} order) Riemann tensor;

$$R = g^{ij} R_{ij} \qquad (1.9)$$

is the so called *"scalar curvature"*;

$\varkappa$ = a constant related to the universal gravitational constant and

T_{ij} = the energy and momentum tensor (an equally weird quantity).

Equation (1.7) is an extremely complex entity. It amounts to 10 non-linear 2^{nd} order differential equations and their solution can only be envisaged after linearization (weak gravitational fields). Furthermore, the resulting equations do not take into account the "state" of the system relating the density with the pressure and any relationship between (1.7) and Maxwell equations still remains unproved ([3], p.87).

It is also worth mentioning that the Lorentz transformations which lay at the very root of the Special Theory of Relativity do not apply in the General Theory. Both theories (Special and General) are therefore to some extent unconnected. Furthermore, none of them has improved its "status" since their announcement at the beginning of the century. Naturally enough, the theories in question, hotly praised by ones, have been bitterly criticized by others. Several alternatives to Lorentz equations have also been (unsuccessfully) proposed [4],[5], whereas the General Theory of Relativity has been prudently left untouched and mostly unexplored because of its intrisic "unexplorability".

In [1], another attempt has been made to tackle with the Special Theory of Relativity. To bypass the difficulties detected in Einstein's theories, the Lorentz equations have been réformulated, in the referenced paper, in the differential form and the basic relationships have been "translated" into the euclidean 4-space, relabelled E-4D. In the present contribution, the new approach taken on in [1] is extended to the motion of particles subject to external forces (accelerations). The forces in question can be either of gravitational or electromagnetic nature. Based on the such a restatement of the relativistic Dynamics (General Theory of Relativity) some unusual results are derived (as it may only be expected).

II. GENERALIZATION OF LORENTZ EQUATIONS.

In [1], the relativistic 4-space, E-4D, has been defined as an euclidean space [6], characterized by two special variables: a scalar, $\psi = \tanh^{-1}(v/c)$, v being the velocity of the particle (or of the whole system) in a given point and refered to a certain (inertial) system, and c the speed of light in the vacuum - and ξ. The latter is the arc parameter and can be identified with the relativistic "0-dimension" axis, $x^0 = \chi$, component of arc.

Let $d\rho = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$ be the elementary radius-vector (in the usual notation) in the ordinary space, refered to the orthonormal (cartesian) coordinate system. Then, according to eq.'s (2.2c) and (2.2d) in [1], the following relationships among $d\rho$, $d\xi$, ψ and the element dt of time hold:

$$\frac{d\rho}{\sinh\psi} = d\xi \qquad (2.1a)$$

$$\frac{dt}{\cosh\psi} = \frac{d\xi}{c} \qquad (2.1b)$$

In [1], Sec.IV, the 4-velocity vector components, (u_ξ, u_x, u_y, u_z) have also been introduced. Here, they will be rewritten in a form similar to (4.6) in [1], although allowing, for generality, the coefficients r_x, r_y, r_z to be functions of both, ξ and ψ. Thus, by definition:

$$u^0 \equiv u_\xi = \frac{c}{\cosh\psi} \qquad (2.2a)$$

and

$$u^1 \equiv \frac{dx}{dt} = c\,\frac{r_x(\psi,\xi)}{\cosh\psi}$$

$$u^2 \equiv \frac{dy}{dt} = c\,\frac{r_y(\psi,\xi)}{\cosh\psi} \qquad (2.2b)$$

$$u^3 \equiv \frac{dz}{dt} = c\,\frac{r_z(\psi,\xi)}{\cosh\psi}$$

On the other hand, the discussion carried out with respect to u^k in [1] also holds in the general case and leads to the relationships:

$$\sum_{\nu=1}^{3} (r^\nu)^2 = \sinh^2\psi \qquad (2.3a)$$

$$\sum_{k=0}^{3} (u^k)^2 = c^2 \qquad (2.3b)$$

In what follows, an orthonormal, coordinate system $(x^0,x^1,x^2,x^3) \equiv (\chi,x,y,z)$ will be used when possible, although, more frequently, we shall resort for convenience to (orthogonal) curvilinear systems (cf. [1], Sec.V). Thus, the distinction between the contravariant and covariant components of vectors becomes essential.

With all that in mind, eq.'s (2.2) will be summarized by writing:

$$u^k = c\,\frac{r^k(\psi,\xi)}{\cosh\psi} = (u^0,u) \qquad (2.4)$$

$r^0 = r_\xi = 1$; u^0 is a scalar and u a normal (3-dimensional) space vector. The last term in (2.4) stands for the usual 4-vector notation [2].

Equations (2.4) are general and, to derive the relativistic law of motion, they only need to be differentiated, with the result:

$$du^k = \frac{c^2}{\cosh\psi} R^k_\xi\, d\xi + \frac{c^2}{\cosh\psi}(R^k_\xi - r^k \tanh\psi)d\psi \qquad (2.5)$$

where R_ξ and R_ψ are two coefficientsd to be determined in each situation. It has to be observed that, in the relativistic space E-4D, the differentials dx^ν ($\nu = 1,2,3$) are not independent of ξ, which means that the unit vectors along the axes x,y,z change (in amplitude, although not in direction) with the speed of the system (refered to another – considered as "fixed" – frame). Furthermore, when taking into account that in the preceding equations the variables are r^k, ξ being the parameter, one arrives at the conclusion that R^k_ξ must be computed using covariant derivatives [5]. Thus, by calling D(.) the intrinsic differenciation operator and because, by definition, $r^\nu = dx^\nu/d\xi$, we have:

$$R^k_\xi = \frac{D(r^k)}{d\xi} = \frac{\partial^2 x^k}{\partial\xi^2} + \Gamma^k_{\alpha\beta}\, r^\alpha r^\beta \qquad (2.6)$$

α,β running through 1,2,3. The similarity of the right member in (2.6) with the left one in (1.6), describing a geodesics, will be commented later on. It is possible for r^ν to vary with $\psi = \psi(\xi)$. In this case,

$$R^k_\psi = \frac{\partial r^k}{\partial\psi} \qquad (2.7$$

Otherwise $R^k_\psi = 0$.

Equations (2.5) and (2.1a) give immediately the components of the 4-acceleration vector:

$$a^k = \frac{du^k}{dt} = \frac{c^2}{\cosh^2\psi}\left[R^k_\xi + (R^k_\psi - r^k \tanh\psi)\,\frac{d\psi}{d\xi}\right] \qquad (2.8)$$

On the other hand, it has been found in [1] (see [1], eq. (6.3)) an expression for the force, generalized over the relativistic space and independent of the reference frame:

$$F_1(\xi) = \frac{c^2 m}{\cosh^2\psi}\,\frac{d\psi}{d\xi} \qquad (2.9)$$

where m stands for the relativistic-transformation invariant (former m_o) mass.

Combining (2.8) and (2.9) we find:

$$a^k = \frac{c^2}{\cosh^2\psi} R^k_\xi + \frac{F_1(\xi)}{m}(R^k_\psi - r^k \tanh\psi) \qquad (2.10)$$

Let $p^k = m\,v^k$ be the particle (or system) momentum. Then, by the general Dynamics, the components of a force in E-4D are:

$$\frac{dp^k}{dt} = ma^k = m\left[\frac{c^2}{\cosh^2\psi} R^k_\xi + \frac{F_1(\xi)}{m}(R^k_\psi - r^k \tanh\psi)\right] \qquad (2.11)$$

System (2.11) will be regarded as the **fundamental equations of the Relativistic Dynamics**, contrasting by their straightforwardness and simplicity with Einstein field equations (1.8). In addition, and as the most important fact, it has to be noticed that the system (2.11) consists of 4 first order, differential equations. And, to solve them, no additional (constraint) conditions are required.

In what follows, system (2.11) will be applied to a number of specific problems. In each case, a meaningful solution will be found and it can also be proved that (2.11) is compatible with Maxwell equations. Thus, the system in question expresses the general law of motion of both mechanical and electromagnetic systems. However, before starting with the applications, some more attention will be devoted to the meaning of the relativistic force $F_1(\xi)$ and its components.

III. COMPONENTS OF THE RELATIVISTIC FORCE.

For simplicity, we will consider again a rectilinear, x-directed motion, with $r_y = r_z = 0$, $r_x = \sinh\psi$ and $R^1_\xi \equiv \partial r_x/\partial\xi = 0$, $R^1_\psi \equiv \partial r_x/\partial\psi = \cosh\psi$. Then, according to (2.10), there are two non zero components of the 4-acceleration vector:

$$a_x = \frac{F_1(\xi)}{m}\left[\cosh\psi - \frac{v_x \sinh\psi}{c}\right] \qquad (3.1a)$$

$$a_\chi = -\frac{F_1(\xi)}{mc}\, v_\chi \sinh\psi \qquad (3.1b)$$

We also have:

$$v_x = c\tanh\psi;\quad v_\chi = c\left[1-\tanh^2\psi\right]^{1/2} = \frac{c}{\cosh\psi} \qquad (3.2)$$

and, after substitution in (3.1), it gives:

$$a_x = \frac{F_1(\xi)}{m \cosh \psi} = \frac{F_{1x}}{m} \qquad (3.3a)$$

$$a_\chi = - \frac{F_1(\xi) \tanh \psi}{m} = \frac{F_{1\chi}}{m} \qquad (3.3b)$$

where the definitions (6.11) in [1] have been used, namely

$$F_{1x} = \frac{F_1(\xi)}{\cosh \psi} \; ; \; F_{1\chi} = - F_1(\xi) \tanh \psi \qquad (3.4)$$

The foregoing results are in agreement with the conclusions of Sec. VI in [1].

According to the results derived in [1], Sec. VI, force $F_1(\xi)$ is orthogonal to the trajectory at each point (event) in E-4Σ (in our case, χ - x plane). Let α be the angle between the tangent to the trajectory and the reference axis χ, at a point P, as depicted in Fig.1. The abscissa $\chi_o = ct_o$ of P measures (divided

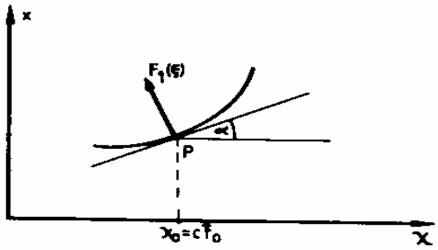

Fig.1. Fragment of atrajectory in the relativistic χ - x plane.

by c) the local time of the system in motion along the trajectory under consideration (cf. [1], Sec.III). The components of the velocity associated with the trayectory on both axes are:

$$v_x = c \sin \alpha = c\, t\, nh\, \psi \qquad (3.5a)$$

$$v_\chi = c \cos \alpha = \frac{c}{\cosh \psi} \qquad (3.5b)$$

in agreement with the α - ψ relationships, established in [1], Sec.II. Then, we can represent the components of force $F_1(\xi)$, refered to in (3.4), in the v_χ - v_x plane. The construction is given in Fig.2, where, for a more easy comprehension of the quantities we are dealing with, other "components" of $F_1(\xi)$ are also shown.

It is clear from the construction that the ensemble of all the x-directed (in the normal space) trajectories lies on the cirumference of radius c (cf.[I], Sec. IV).

Refer again to Fig.1. The length of an infinitessimal arc is:

$$dw = c\, dt = \cosh \psi \;\; d\xi \qquad (3.6)$$

Because $d\chi \equiv d\xi$ (two different meanings assigned to the same quantity), eq.(3.6)

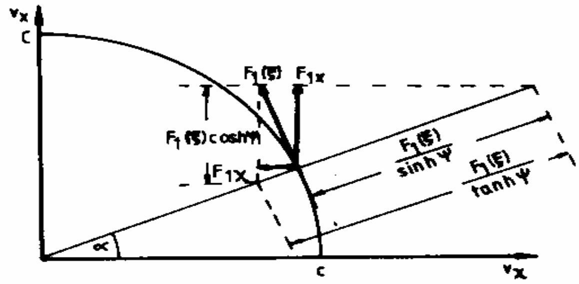

Fig.2. Components of the relativistic force $F_1(\xi)$ in the plane of velocities.

expresses w as a function of χ. Clearly, w is a smooth (for all material systems), non decreasing function of χ. Then, an inverse, smooth, single valued function $\chi = \chi(w)$ also exists and, except for very special circumnstances which will be considered elswhere, $\chi(w)$ is of C" type (i.e. at least twice differentiable). In the ordinary geometry and selecting ξ as parameter, $d^2w/d\xi^2$ represents the curvature of the trajectory [5] for an observer at rest (moving uniformly along χ-axis). On the other hand, for a traveller experimenting an acceleration, it is the inertial reference system that is moving and possibly rotating, as everybody has experimented it by himself (or herself). Then, $d^2\chi/dw^2$ can be interpreted as the **curvature of the space"** - a crucial concept in Einstein General Theory of Relativity.

Alternatively, following the new approach, it is easy to find fr m (3.6 and after the substitution of ξ by χ when needed:

$$\frac{d^2\chi}{dw^2} = \frac{d}{dw}\left[\frac{1}{\cosh^2\psi}\right] = - \frac{\sinh \psi}{\cosh^2\psi} \frac{d\psi}{dw} =$$

$$= \frac{\sinh \psi}{\cosh^2\psi} \frac{d\chi}{dw} \frac{d\psi}{d\xi}$$

or, using again (3.6) and the definition of $F_1(\xi)$:

$$\frac{d^2\chi}{dw^2} = - \frac{F_1(\xi)}{c^2 m} \tanh \psi \qquad (3.7a)$$

At rest, ψ = 0 and **the curvature of the space is exactly zero!** (The same is true when $F_1(\xi)$ = 0, which means the absence of any field of force). At the other extremum, when $\psi \rightarrow \infty$ (the particle velocity approaches the speed of light), we have:

$$\frac{d^2\chi}{dw^2} \rightarrow - \frac{F_1(\xi)}{c^2 m} \qquad (3.7b)$$

that is to say, the curvature is directly proportional to $F_1(\xi)$ and inversely to the particle mass; (and energy). For example, for a photon, c^2m must be replaced by $h\nu$ and, with $F_1(\xi)$ due to a gravitational field, the curvature of the trajectory is finite, the fact already predicted by the *Schwarzschild equation* and, as it stands, confirmed experimentally [2], [3], [7].

IV. RECTILINEAR MOTION.

Equation (2.11) will now be applied to solve some problems of rectilinear motion. It must also be emphasized from the very beginning that there will be no restrictions concerning the speed of the particle or the strength of the applied force, which is not the case with Einstein field equation (1.7), supposedly valid (7) for low velocities and weak forces (only). In our case, let again the motion be oriented along x-axis, the acceleration a_x being given by (3.3a). Because

$$a_x = \frac{dv_x}{dt} = \frac{c}{\cosh \psi}\,\frac{dv_x}{d\xi} \qquad (4.1)$$

we have, using (3.3a) and (4.1):

$$\frac{dv_x}{d\xi} = \frac{F_1(\xi)}{mc} \qquad (4.2)$$

an equation remarkable by its simplicity. It clearly states the Newton 2^{nd} law for a rectilinear motion in an invariant - with respect to the relativistic transformations - form and will be used as the starting point for solving our next problems.

IV-1. Uniform motion.-

The uniform rectilinear motion is characterized by the condition $F_1(\xi) = 0$. Then, by (4.2), $v_x = V$ (constant) and trivially represents the well known free motion of the system, considered in some detail in [1].

IV-2. Uniformly accelerated motion.-

It will be supposed

$$\frac{F_1(\xi)}{m} = a \text{ (constant)}$$

Using (4.2) and the fact that, by definition, $v_x = dx/dt$ and $dt = \frac{d\xi}{c}\cosh\psi$, one readily arrives at the equations:

$$dt = c^{-1}[1-M(\xi)^2]^{-1/2}\,d\xi. \qquad (4.4a)$$

$$dx = M(\xi)\,[1-M(\xi)^2]^{-1/2}\,d\xi \qquad (4.4b)$$

with

$$M(\xi) = \frac{a}{c^2}(\xi - \xi_o) + \frac{v_o}{c} \qquad (4.5)$$

and ξ_o, v_o the initial conditions.

Equations (4.4) are real if $M(\xi) \le 1$. To see the meaning of this constraint suppose $\xi_o = v_o = 0$. Then,

$$a \le c^2/\xi \qquad (4.6)$$

On the other hand, by direct integration of the equation

$$\frac{dv_x}{d\xi} = \frac{a}{c}$$

with the initial conditions as stated, we arrive at:

$$v_x = \frac{a\,\xi}{c} \le c \qquad (4.7)$$

where, again, the condition (4.6) has been used. Because one can always bring the initial conditions to the zero state by conveniently choosing the (inertial) reference system, condition (4.7) is general and simply states the only too well known fact that the velocity of any particle or system cannot exceed the speed of the light.

System (4.4) is easy to integrate after performing the change of variable, allowed by (4.6):

$$M(\xi) = \sin\beta \qquad (4.8)$$

with the result:

$$t - t_o = \frac{c}{a}(\beta - \beta_o) \qquad (4.9a)$$

$$x - x_o = \frac{c^2}{a}(\cos\beta_o - \cos\beta) \qquad (4.9b)$$

and

$$\cos\beta_o = \left[1 - v_o^2/c^2\right]^{1/2} \qquad (4.10)$$

t_o being an arbitrarily chosen origin of time.

It is interesting to observe that in this example we arrive at the classical formula for the uniformly accelerated motion by letting $v/c \to 0$. In effect, in such a case,

$$\cos\beta_o \cong 1 - \beta_o^2/2\ ; \quad \cos\beta \cong 1 - \beta^2/2$$

which gives, after some simple operations:

$$x - x_o = \frac{a}{2}(t - t_o)^2 + \beta_o c(t - t_o) \cong$$

$$\cong \frac{a}{2}(t - t_o)^2 + v_o(t - t_o)$$

because

$$\beta_o \cong \sin\beta_o = \frac{c^2}{a}$$

Now, we shall briefly return to condition (3.6). According to (4.9b), the longest distance, travelled by the object (before reaching the speed of light) is

$$x_{max} = \frac{c^2}{a} \qquad (4.11)$$

Suppose $a = 10g \cong 100\ m/s^2$. It gives:

$$x_{max} = \frac{9 \times 10^{16}}{100} = 9 \times 10^{14}\ m =$$

$$= 0.0957 \text{ light years}$$

It is easily verified that, by classical formula for the uniformly accelerated motion, the maximum distance in the same conditions would be $c^2/2a$, that is to say, the particle would reach the speed of light after travelling exactly half the distance predicted by the relativistic approach.

An interesting question now arises: is such a law of motion possible in a long run ($v_x \rightarrow c$)? Clearly, a force independent of the speed can only be applied internally by an object, such as a rocket or, equivalently, with the source of force moving with the object. But, according to what has been said in [1], Sec. VI, the speed of light can only be achieved by an object after all its mass (including the crew and the passengers inside) had been transformed into the radiating energy. It is not likely that anybody should find such an expedition very attractive.

A more realistic case will be treated next.

IV-3. Motion under the inverse square law.-

In this example, the motion of a particle under the inverse square law attraction will be considered. In such a case, the attracting force is conveniently described by the term:

$$F_{1x} = - \frac{F_1(\xi)}{\cosh \psi} = - \frac{B}{x^2 \cosh \psi} \qquad (4.12)$$

with B a constant. Then, by (4.2):

$$\frac{dv_x}{d\xi} = - \frac{B}{mc\ x^2}$$

After replacing $d\xi$ by $dx/\sinh \psi$ we find:

$$\sin \psi \ \frac{dv_x}{dx} = - \frac{B}{mc\ x^2} \qquad (4.13)$$

On the other hand,

$$\frac{dv_x}{dx} = c \frac{d}{dx} (\tanh \psi) = \frac{c}{\cosh^2 \psi} \ \frac{d\psi}{dx}$$

which, when combined with (4.13) and after some simple manipulation, leads to the equation:

$$- d\left(\frac{1}{\cosh \psi}\right) = \frac{B}{mc^2} d\left(\frac{1}{x}\right)$$

Integrating:

$$\frac{1}{\cosh \psi} - \frac{1}{\cosh \psi_o} = \frac{B}{mc^2} \left(\frac{1}{x_o} - \frac{1}{x} \right) \qquad (4.14a)$$

ψ_o and x_o are the initial values of ψ and x, respectively. If we suppose for simplicity that the particle starts at the infinity with the initial speed zero, that is, $x_o = \infty$, $\psi_o = 0$, eq.(4.14a) can be rewritten more simply as:

$$\frac{1}{\cosh \psi} = 1 - \frac{B}{mc^2 x} \qquad (4.14b)$$

For ψ real, the term $1/\cosh \psi$ is non negative, then,

$$x \geq \frac{B}{mc^2}$$

where the sign (=) holds for $\psi = \infty$ and gives the distance at which the particle acquires the speed of light.

To arrive at the equation of motion we start with the identity:

$$\frac{1}{\cosh^2 \psi} \equiv 1 - \tanh^2 \psi = 1 - \frac{1}{c^2} \left(\frac{dx}{dt} \right)^2$$

which, when combined with (4.14b) and after a slight rearrangement, gives:

$$dt = - \frac{2mc^2 x \ dx}{2cB \left[\frac{2mc^2 x}{B} - 1 \right]^{1/2}} \qquad (4.15)$$

where the sign (−) is due to the initial hypothesis that the particle is attracted towards the origin and, thus, the distance decreases with the time.

The integration of (4.15) is straightforward, giving:

$$t - t_o = - \frac{B}{3mc^3} \left[\frac{2mc^2 x}{B} - 1 \right]^{1/2} \left[\frac{mc^2 x}{B} + 1 \right] + \text{Const.} \qquad (4.16)$$

The integration constant depends on the selection of the origin of time, t_o.

Suppose, for instance, that the particle under consideration is an electron attracted by a proton. The attracting force vanishes when the electron acquires the speed of light with respect to the proton, in which case the former, converted into radiation energy plus the corresponding negative charge, impinges on the former. The net result is the well known phenomenon of capture of an electron by a proton creating a neutron (the intervention in this process of other, "strange" particles, such as neutrinos to assure the momentum conservation, postulated by theoretical physicists, does not impair the foregoing reasoning). Under the previous suppositions, we can choose the origin of time as the instant of the electron capture, that is, when x reaches its minimum:

$$x_n = \frac{B}{mc^2} \quad \text{for} \quad t = t_o \qquad (4.17)$$

Then, by (4.16):

$$\text{Const.} = \frac{2B}{3mc^3}$$

and

$$t - t_o = \frac{2B}{3mc^3} \left\{ 1 - \frac{1}{2} \left[\frac{2mc^2 x}{B} - 1 \right]^{1/2} \left[\frac{mc^2 x}{B} + 1 \right] \right\} \qquad (4.18)$$

Equation (4.18) can be used, for example, for computing the time-of-flight of the particle before being captured.

In the specific case of capture of an electron, we have in IS (International System) units

$$B = \frac{q_e^2}{4\pi\varepsilon_o} \qquad (4.19)$$

(q_e = charge of the electron = $= 1.6 \times 10^{-19}$ Coul., $\varepsilon_o = \frac{10^7}{4\pi c^2}$ F/m).

Then,

$$x_n = \frac{1}{4\pi\varepsilon_o} \frac{q_e^2}{mc^2} \qquad (4.20a)$$

or, numerically:

$$x_n = 0.2818 \times 10^{-14} \text{ m} \qquad (4.20b)$$

which is, in effect, of the same order as the (estimated) radius of the neutron (approximately twice the radius of a proton [8]). It is possible to compare equation (4.20a) with the *gravitational* radius [2],[3],[7],[10]:

$$r_g = \frac{2km}{c^2} \qquad (4.21a)$$

where k is the gravitational constant. To this end, we notice that, by calling F_g and F_c the gravitational force and the coulomb force, respectively, in IS units, one finds the relationship:

$$\frac{F_g}{F_c} = 4\pi\varepsilon_o k \frac{m^2}{q_e^2}$$

Furthermore, using (4.20a) and (4.21a), we get;

$$\frac{r_g}{x_n} = \frac{2km^2 4\pi\varepsilon_o}{q_e^2}$$

For the sake of comparison, let $F_g = F_c$, with the result

$$r_g = 2x_n \qquad (4.21b)$$

a shocking result showing that there is stil a connection between the old theory and the new one.

It is also worth while to observe that, if we denote by a_o the mean radius of the first Bohr orbit [8]:

$$a_o = \frac{h^2}{4\pi m_e q_e} = 0.529 \times 10^{-10}\text{m} \qquad (4.22)$$

(m_e = mass of the electron), then the following relationship holds:

$$\frac{x_n}{a_o} = 5.327 \times 10^{-5} = \alpha_f^2$$

with

$$\alpha_f = \frac{1}{2\varepsilon_o} \frac{q_e^2}{ch} \cong \frac{1}{137} \qquad (4.23)$$

the famous *fine structure constant* of the atom. Also, by combining (4.19) and (4.23) we find:

$$B = \frac{ch}{2\pi} \alpha_f \qquad (4\text{-}24)$$

Evidently, the computations of this subsection remain valid (with only changing the sign of dx) in the inverse process, i.e., the decay of the neutron by emitting a particle beta. Here, we assume that the latter is endowed with the initial velocity equal (or almost equal) to c. Evidently, this is a quite new theory and has to be confronted with the current understanding of this kind of nuclear reactions. More about the behaviour of electrons is to be said in what follows and elswhere.

V. CENTRAL-FORCE MOTION.

V-1. General statement of the problem.-

The next in order of complexity after the rectilinear motion and the last to be treated here is the orbital motion under the effect of a central force. The force in question will be assumed to be emanating from the centre O and actuating on the point P (see Fig.3), orbitting with the speed v along the trajectory τ. We shall call ρ the radius vector of P, v_ρ and v_η will be the radial and normal components, respectively, of the velocity of P. In the analysis that follows, the vectors (in the 3-dimensional euclidean space) will be labelled by lower-case bold-face letters and the vector amplitudes by the same, normal lightface, letters. The motion we are considering here is sufficiently close to the general one for some rather interesting - and not all of them quite usual - consequences to be derived.

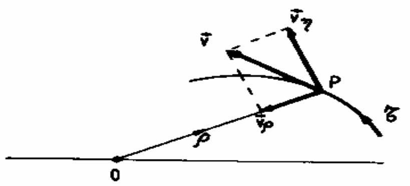

Fig.3. Components of the velocity of an orbitting object, actuated upon by a central force.

From Classical Dynamics we know that the amplitudes of the radial and normal velocity components are:

$$v_\rho = \frac{d\rho}{dt} ; \quad v_\eta = \rho \frac{d\eta}{dt} \qquad (5.1)$$

Some more remarks about the notation to be used are needed now. The angle between the radius vector and the reference direction has been labelled η in order to spare the more common denomination ϕ for later uses. On the other hand, vector components, marked with subscripts - mostly using greek low-case letters - will denote the metric components [9]. The super-scritpts will be given the meaning of contravariant indices of tensors (when alphabetic) or of powers (when numeric). Repeated (dumb) alphabetic cross-indices will (generally) obey Einstein's summation rule.

Assembling all these results and after eliminating $F_1(\xi)$ with (2.9) we get:

$$a_\rho = a^k\Big|_{k=1} = \frac{c^2}{\cosh^2\psi}\left[\frac{d^2\rho}{d\xi^2} - \rho\left(\frac{d\eta}{d\xi}\right)^2 - \frac{d\rho}{d\xi}\frac{d\psi}{d\xi}\tanh\psi\right] \quad (5.7a)$$

$$a_\eta = \rho a^k\Big|_{k=2} = \frac{c^2}{\cosh^2\psi}\left[\rho\frac{d^2\eta}{d\xi^2} + 2\frac{d\rho}{d\xi}\frac{d\eta}{d\xi} - \rho\frac{d\eta}{d\xi}\frac{d\psi}{d\xi}\tanh\psi\right] \quad (5.7b)$$

(a^k are the contravariant components of the 4-acceleration vector). After performing some further rearrangement, we arrive at the result:

$$a_\rho = \frac{c}{\cosh\psi}\frac{d}{d\xi}\left[\frac{c}{\cosh\psi}\frac{d\rho}{d\xi}\right] - \frac{c^2}{\cosh^2\psi}\rho\left(\frac{d\eta}{d\xi}\right)^2 \quad (5.8a)$$

$$a_\eta = \frac{1}{\rho}\frac{c}{\cosh\psi}\frac{d}{d\xi}\left[\rho^2\frac{c}{\cosh\psi}\frac{d\eta}{d\xi}\right] \quad (5.8b)$$

Returning to the definition of the velocity in the relativistic (E-4D) space, and taking into account that v_ρ and v_η are mutually orthogonal, we have from (5.1):

$$v^2 = c^2\tanh^2\psi = \left(\frac{d\rho}{dt}\right)^2 + \left(\rho\frac{d\eta}{dt}\right)^2 \quad (5.2)$$

It is also well known that the components of an acceleration in the polar coordinates (ρ,η) are (see any standard book on Mechanics and also APPENDIX A):

$$a_\rho = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\eta}{dt}\right)^2 \quad (5.3a)$$

$$a_\eta = \frac{1}{\rho}\frac{d}{dt}\left(\rho^2\frac{d\eta}{dt}\right) \quad (5.3b)$$

It is easy to prove that equations (5.3) are the special case of the general relativistic motion equations (2.5) and (2.6). In effect, the only non zero components of Cristoffel symbols in polar coordinates are (see APPENDIX A):

$$\Gamma^\rho_{\eta\eta} = -\rho\ ; \quad \Gamma^\eta_{\rho\eta} = \Gamma^\eta_{\eta\rho} = \frac{1}{\rho} \quad (5.4)$$

the values of the (normalized) 4-velocity vector r^k being

$$r^k\Big|_{k=1} = r^\rho = \frac{d\rho}{d\xi}\ ; \quad r^k\Big|_{k=2} = r^\eta = \frac{d\eta}{d\xi} \quad (5.5)$$

Then, by combining (5.5) with the method outlined in APPENDIX A, the following relationships are derived [see also (2.6)]:

$$R^\rho_\xi = \frac{d^2\rho}{d\xi^2} - \rho\left(\frac{d\eta}{d\xi}\right)^2 \quad (5.6a)$$

$$R^\eta_\xi = \frac{d^2\eta}{d\xi^2} + \frac{2}{\rho}\frac{d\rho}{d\xi}\frac{d\eta}{d\xi} \quad (5.6b)$$

On the other hand:

$$R^\rho_\psi = R^\eta_\psi = 0$$

because neither ρ nor η depend explicity on ψ (as it will be proved later on).

and (5.3) follows.

V-2. Equation of motion.-

From the relativistic standpoint, the classical motion conditions:

$$a_\rho = \frac{K}{m\rho^2}\ ; \ a_\eta = 0$$

must be replaced by

$$a_\rho = \frac{F_1(\xi)}{m\cosh\psi} = \frac{K}{m\rho^2\cosh\psi} \quad (5.9a)$$

$$\frac{d}{d\xi}\left(c\rho^2\frac{d\eta}{d\xi}\right) = 0 \quad (5.9b)$$

The first one is due to the definition of F_{1x} given in (3.4) and the second owing to the fact that a_η is orthogonal to ρ. Thus, the radial component of v_η is zero and the derivative in (5.8) must be evaluated assumming ψ constant. It also means that (5.3b) has to be computed in terms of the rest time, measured with respect to the reference frame associated with the central system, in agreement with the 2nd Kepler law.

After integrating (5.9b) we have:

$$c\rho^2\frac{d\eta}{d\xi} = B \quad \text{(constant)}$$

that is,

$$\frac{d\eta}{dt} = \frac{B}{\rho^2\cosh\psi} \quad (5.10)$$

Equations (5.10) and (5.9a), when combined with (5.3a), give the law of motion (see APPENDIX B) under the central-force conditions:

$$\frac{d^2p}{d\eta^2} + p = -\frac{K \cosh \psi}{mB^2}\left[1 - \frac{B \sinh \psi}{c}\frac{dp}{d\eta}\right] \qquad (5.11)$$

with $p = 1/\rho$.

V-3. Approximate solution of the orbital motion.-

Before attempting to find the exact solution of (5.11) the last term in (5.11) will be supposed negligible (it is always true, unless ψ or $dp/d\eta$, or both, are sufficiently large, which conditions will be considered later on). The approximate equation to be solved is therefore:

$$\frac{d^2p}{d\eta^2} + p = -\frac{K \cosh \psi}{mB^2} \qquad (5.12)$$

On the other hand, by replacing (B2) from APPENDIX B into (5.2) it is found:

$$\left(\frac{dp}{d\eta}\right)^2 + p^2 = \frac{c^2 \sinh^2\psi}{B^2} \qquad (5.13)$$

After differentiating, dividing by 2 and some rearrangements we get:

$$\frac{d^2p}{d\eta^2} + p = \frac{c^2}{B^2}\sinh\psi \cosh\psi \frac{d\psi}{d\eta}$$

which, when combined with (5.12), gives:

$$\frac{dp}{d\psi} = -\frac{mc^2}{K}\sinh\psi \qquad (5.14)$$

Let $\psi_\infty = \psi\big|_{p=0}$

Then, by integrating (5.14), we easily arrive at:

$$\cosh\psi = -\frac{K}{mc^2}p + \cosh\psi_\infty \qquad (5.15)$$

and, from (5.12),

$$\frac{d^2p}{d\eta^2} + \left[1 - \frac{K^2}{m^2B^2c^2}\right]p + \frac{K}{mB^2}\cosh\psi_\infty = 0 \qquad (5.16)$$

which is a linear differential equation, easy to solve. To do that, let

$$\gamma^2 = 1 - \frac{K^2}{m^2B^2c^2} = 1 - \delta^2$$

By adequately choosing the origin of phases the solution of (5.16) can be written as:

$$p = -\frac{K\cosh\psi_\infty}{m\gamma^2B^2}\left[1 - \frac{m^2\gamma^2B^2}{K\cosh\psi_\infty}A\cos(\gamma\eta)\right] \qquad (5.17)$$

which is, as it is well known from Classical Mechanics (see also [8]), the equation of a conic with the excenticity:

$$\epsilon = \frac{m\gamma^2B^2}{K\cosh\psi_\infty}A \qquad (5.18)$$

Constant B is easily found by applying the angular momentum conservation law. Thus, from (5.10) we find

$$mv_\eta\rho = \frac{B\,m}{\cosh\psi}$$

and, for example in the case of an electron, it amounts, according to the 1^{st} Bohr axiom [8], to kh/2, with k an integer. Thence:

$$B = \frac{kh}{2\pi m\cosh\psi} \qquad (5.19)$$

The integration constant A and, from this, the orbit excentricity can be found by applying the energy conservation law. In the Relativistic Mechanics, it is best written in the form (cf. [1], eq. (6.8)):

$$\frac{mv^2}{2} = \frac{mv_\infty^2}{2} + \int_\infty^\rho F_\rho\, d\rho$$

with $F_\rho = K/\rho^2 \cosh\psi$. The computation is easily carried out by eliminating $\cosh\psi$ with (5.15) and is of no particular interest for the present discussion.

Of a real importance is however the factor $\gamma < 1$ multiplying the angle η in (5.17). It states that the radius-vector ρ in the case of an elliptic orbit needs to run an angle

$$\Delta\eta = 2\pi\left(\frac{1}{\gamma} - 1\right) \qquad (5.20)$$

radians in excess of 2π, before recovering its initial value. In the case of an electron orbit it leads to the Sommerfeld correction of the Bohr orbit [8]:

$$\gamma = \left[1 - \frac{Z^2q_e^2}{cp_a^2}\right]$$

with Z the atomic number and p_a the angular momentum.

It is also to be noted that, for the attracting force between a proton and an electron, and using again IS units, we have:

$$K = -\frac{q_e^2}{4\pi\epsilon_o} = -2.310 \times 10^{-28}$$

$$B = \frac{h}{4\pi m_e} = 1.158 \times 10^{-4}$$

$$\delta = \frac{K}{mBc} = \frac{q_e^2}{2\epsilon_o ch} = \alpha_f$$

with $\alpha = 1/137$, the fine structure constant of atoms, already encountered in (4.23).

For planetary motions, however, the value of γ given by eq. (5.16) does not agree either with the experimentally measured results or those predicted by *Schwarzschild* equation [2],[3],[7],[10] under idealized conditions of a static central-force field with spherical symmetry. For example, for the orbit of Mercury, the masured lag of the perihelion is about 43'' in a century. On the other hand, the value predicted by the Schwarzschild equation for the same quantity is ≈ 41'', whereas the lag computed from (5.20) using the known parameters of Mercury orbit accounts only for 7.13''. The difference is quite significant and can only be justified by assuming that the conditions under which the perihelion lag has been computed in our case are strongly oversimplified. Evidently, the subject needs further investigation

V-4. Comments on the exact solution of the central force problem: electric circuit model of orbital motion.-

When the term $\frac{B}{c}\sinh\psi\frac{dp}{d\eta}$ is no longer vanishingly small as compared with the unity, the problem of central-force motion becomes more complex. Some advantage may be gained by resorting to the "state-space" method [11],[12]. To this end, two "state-space variables" can be defined as:

$$p_1 = p \; ; \quad p_2 = \frac{dp}{d\eta}$$

leading to the decomposition of eq.(5.11) into the "space-state equatios":

$$\frac{dp_1}{d\eta} = p_2 \qquad (5.21)$$

$$\frac{dp_2}{d\eta} = -p_1 + \frac{K\sinh\psi\ \cosh\psi}{mcB} - \frac{K\cosh\psi}{mB^2}$$

together with the *"connecting equation"*

$$p_1^2 + p_2^2 = \frac{c^2\sinh^2\psi}{B^2} \qquad (5.22)$$

System (5.21)-(5.22) can be solved by any standard numerical iterative routine, provided the initial conditions $[p_1(0),\ p_2(0),\ \psi(0)]$ are adequately chosen.

Alternatively, the system in question can be simulated by an electric circuit, schematically represented by the *"flow graph"* [12], [13] of Fig. 4, which operates as follows.

A central nucleus of the system (enclosed in the dotted-line box) contains two differentiation operators, D and -D (without and with the sign inversion, respectively). Thus, in the upper branch of the nucleus, operator -D, actuating on p_1, generates $-p_2$, according to the first equation (5.21). Likewise, the lower branch delivers to "node" *g* the derivative of $-p_2$, namely $-dp_2/d\eta$.

In a "normal" operation of the system depicted in Fig.4, switch *S* is closed simulating the (non vanishing) stabilizing contribution of the second member in eq. (5.11). Then, following the ascending paths outside the nucleus, both variables, p_1 and p_2, are squared by the operators x^2. In *a*, both squares are added and the sum is multiplied in *b* by the factor B^2/c^2 delivered from outside as an external "signal". According to eq. (5.11), it creates the function $\sinh^2\psi$ which, when added to the unity in *d*, propagates downward, along the left-side branch, as $\cosh^2\psi$. The operator labelled $\sqrt{x}$ takes the square root of the oncoming signal which, after being multiplied in *e* by $-K/mB^2$, generates the last term in the second equation (5.21). Likewise, it is not difficult to see that the right-hand channel delivers to node *f* the next to last term in the last equation. Both signals, incoming from the left-side and right-side channels, are added to $-dp_2/d\eta$ in *g*, generating p_1.

Summing up the whole procedure, the system depicted in Fig.4, with *S* closed, works as a freely oscillating nucleus (the homogeneous part), stabilized by the second member (the forcing part) of eq.(5.11). The contribution of the right-hand channel is usually negligible.

However, a sudden change in operating conditions may originate a large increase in the variable $p_2 = dp_1/d\eta$ which

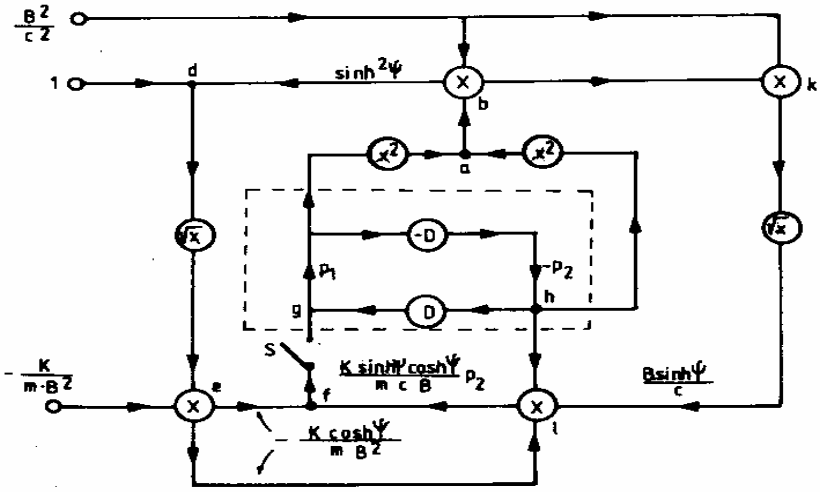

Fig.4. Electric circuit model of the orbital motion.

tends to neutralize the stabilizing contribution of the left-hand channel. A complete neutralization [cancellation of the second member in (5.11)] amounts to the opening of the switch *S* and leaves the system free to uncontrollably "run away", until p_z sufficiently decreases and forces the simulated circuit to become stabilized in another "state". Note that the constants (input variables) *K* and *B*, confer two degrees of freedom to the device, allowing the accommodation of a wide range of masses and variables p. In particular, the electron transitions (for example under the impacts of photons) or nuclear transmutations due to sudden (and usually very brief) forces (Dirac impulses-like functions) may be simulated by the instantaneous opening of S. In any case, the circuit depicted in Fig.4 and the present duscussion of its operation clearly shows the difficulty, if not the outright impossibility, to outline the general solution of eq (5.11) because of its dramatic dependence on the actuating forces. This conclusion is in agreement with the modern catastrophic theory [14], concerning the system characterized by non linear differential equations.

In extreme situations of very strong accelerations which eventually lead to unstabilities and transitions or "jumps" to other states, the equations of determninistic mechanics no longer hold. Particles - or matter in general - abide under such conditions by other laws, possibly defined in other spaces, presently under active investigation in *Quantum Mechanics* and *Group Theory*. These parallel branches of research on the behaviour of the Nature are clearly beyond the scope of the present paper. What has to be pointed out here is the fact that the proposed equations apply to both mechanical objects and electromagnetic systems, as it has been illustrated by the foregoing examples, and will be pursued in more detail elswhere.

APPENDIX A

Let a(s) be a vector associated with a point *P*(s) on a trajectory τ in arbitrary curvilinear coordinates (see Fig.A1), a(s+ds) is the same vector displaced a vanishing small distance ds (along τ) to *P*(s+ds). The difference

$$da = a(s+ds) - a(s) \qquad (A.1)$$

has no definite physical meaning because both vectors in the second member of (A.1) are attached to different points in space (it is also proved [2], [10], that it does not abide by the tensor transformation law either). To regain a vector in the usual sense we form a new differential element

$$Da = da - \delta a \qquad (A.2)$$

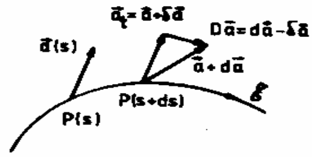

Fig.A1.Translation and differiation of a vector in curvilinear coordinates.

where δa is the difference between the result of the parallel transportation of a(s) to *P*(s+ds) and the original vector a(s), as illustrated in Fig. A1. Da, defined in (A.2), is the so called "intrinsic differential" of a(s) [10].

In a linear space [6], the increment δa^i due to parallel displacement of the (contravariant, to be specific) component a^i of a can be written in the form:

$$\delta a^i = - \Gamma^i_{jk}\, a^j\, dx^k \qquad (A.3)$$

Γ^i_{jk} being Christoffel symbols of 2^{nd} kind[1] and the sign (-) in the 2^{nd} member has been introduced for convenience. On the other hand, the relationship

$$da^i = \frac{\partial a^i}{\partial x^k}\, dx^k$$

evidently holds. Then, after combining (A.2) and (A.3) we get

$$Da^i = \left(\frac{\partial a^i}{\partial x^k} + \Gamma^i_{jk}\, a^j\right) dx^k \qquad (A.4)$$

A typical example of a contravariant vector is the velocity vector [2], [9]. In polar coordinates, applying (A4) to

$$v^k\big|_{k=1} = v^\rho ; \quad v^k\big|_{k=2} = v^\eta$$

(contravariant componeents, not to be mistaken for the metric ones) we have:

$$\begin{aligned} Dv^\rho &= \left(\frac{\partial v^\rho}{\partial \rho} + \Gamma^\rho_{\rho\rho} v^\rho\right) d\rho + \Gamma^\rho_{\eta\rho}(v^\rho d\eta + v^\eta d\rho) + \Gamma^\rho_{\eta\eta} v^\eta d\eta \\ Dv^\eta &= \frac{\partial v^\eta}{\partial \eta}\, d\eta + \Gamma^\eta_{\rho\rho} v^\rho d\rho + \Gamma^\eta_{\rho\eta}(v^\rho d\eta + v^\eta d\rho) + \Gamma^\eta_{\eta\eta} v^\eta d\eta \end{aligned} \qquad (A.5)$$

where the indices ρ and η are particular values of *i*,*j*,*k*; thus Einstein's summation convention does not hold and the well known symmetric properties of Christoffel symbols with respect to the lower indices [7], [10] have been used.

Let ξ be a parameter (possibly the time). Then, it is well known by other considerations (see any standard book of Mechanics and also [8]) that the (metric) components of acceleration in polar coordinates are

$$\begin{aligned} a_\rho &= \frac{Dv^\rho}{d\xi} = \frac{d^2\rho}{d\xi^2} - \rho\left(\frac{d\eta}{d\xi}\right)^2 \\ a_\eta &= \rho\,\frac{Dv^\eta}{d\xi} = \rho\,\frac{d^2\eta}{d\xi^2} + 2\,\frac{d\rho}{d\xi}\,\frac{d\eta}{d\xi} \end{aligned} \qquad (A.6)$$

Then, by substituting (A.5) into (A6) and using the definition of the contravariant components of the velocity vector in polar coordinates: $v^\rho = d\rho/d\xi$, $v^\eta = d\eta/d\xi$, we easily arrive at the conclusion that the only non zero components of Γ^i_{jk} are [a]

$$\Gamma^{\rho}_{\eta\eta} = -\rho; \quad \Gamma^{\eta}_{\eta\rho} = \Gamma^{\eta}_{\rho\eta} = \frac{1}{\rho} \qquad (A.7)$$

APPENDIX B

The proposed problem is to derive the law of motion for the system characterized by the equations:

$$\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\eta}{dt}\right)^2 = \frac{K}{m\rho^2\cosh\psi} \qquad (B.1a)$$

$$\frac{d\eta}{dt} = \frac{B}{\rho^2\cosh\psi} \qquad (B.1b)$$

Following the classical method [8], it is easily done by performing the change of variable $\rho = 1/p$, with the result:

$$\frac{d\rho}{dt} = \frac{d\rho}{d\eta}\frac{d\eta}{dt}; \quad \frac{d\rho}{d\eta} = -\frac{1}{p^2}\frac{dp}{d\eta}$$

By (B.1b):

$$\frac{d\eta}{dt} = \frac{Bp^2}{\cosh\psi} \qquad (B.2a)$$

Then we have

$$\frac{d\rho}{dt} = -\frac{B}{\cosh\psi}\frac{dp}{d\eta} \qquad (B.2b)$$

Differentiating again, the following succession of equations is readily found:

$$\frac{d^2\rho}{dt^2} = B\frac{\sinh\psi}{\cosh^2\psi}\frac{d\psi}{dt}\frac{dp}{d\eta} - \frac{B}{\cosh\psi}\frac{d^2p}{d\eta^2}\frac{d\eta}{dt} =$$

$$= \frac{cB\sinh\psi}{\cosh^2\psi}\frac{d\psi}{d\xi}\frac{dp}{d\eta} - \frac{B^2p^2}{\cosh^2\psi}\frac{d^2p}{d\eta^2}$$

$$= \frac{B\sinh\psi}{\cosh\psi} - \frac{F_1(\xi)}{cm}\frac{dp}{d\eta} - \frac{B^2p^2}{\cosh^2\psi}\frac{d^2p}{d\eta^2}$$

$$(B.3)$$

Because, according to (5.9a), $F_1(\xi)/m = Kp^2$, then, by replacing (B.3) and (B.2a) into (B.1a) and rearranging terms, we get (5.11).

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[1]) Some authors [7] call $-\Gamma^i_{jk}$ *affine connections* and assign different symbols to the Christoffel "operators".

[a]) Alternatively, (A7) can be computed from the relationschip between the componentes of the metric tensor and the Christoffel symbols [2], [10]. Then, components (A6) of the acceleration can be easily derived.

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ABOUT THE AUTHOR

(See "RELATIVE MOTION IN EUCLIDEAN SPACE AND TIME"), this Issue.