SADDLE-NODE BIFURCATIONS IN POWER SYSTEMS

Claudio A. Cañizares
Escuela Politécnica Nacional

Abstract: Saddle-node bifurcations are dynamic instabilities of differential equation models that have been associated to voltage collapse problems in power systems. This paper presents the conditions needed for detecting these type of bifurcations in a dynamic model of ac/dc systems, represented by differential equations and algebraic constraints, using power flow equations. It is also shown that two methods typically used to detect saddle-node bifurcations, namely, direct and parameterized continuation methods, are numerically robust at the bifurcation point, which makes them ideal for computer implementation.

1 Introduction

Voltage collapse problems in power systems are generally associated to saddle-node bifurcations as discussed in several papers [1, 2, 3, 4]. These type of instabilities are usually local bus voltage problems due to lack of reactive power support, and can be characterized by a sudden voltage drop with a system wide loss of stability.

Saddle-node bifurcations are well defined instabilities in power system models fully represented by differential equations [1, 5, 6, 7]. However, bifurcations in systems that are also modeled with algebraic constraints have not been thoroughly studied. This paper addresses this problem, showing the requirements for having equivalency of typical saddle-node bifurcation conditions between differential equation models and mixed models, i.e., systems represented by both differential equations and algebraic constraints. Furthermore, it is also shown the conditions for using power flow equations, which in power systems analysis usually yield a good first approximation to system equilibria, to detect saddle-node bifurcations in the proposed full ac/dc dynamic system model. This subject has been of some concern in power system analysis as shown in [5, 8, 9].

Finally, the paper demonstrates, based on saddle-node bifurcation conditions, the robustness of two methods, i.e., direct and parameterized continuation methods, used to detect these type of bifurcations. These methods have been shown to be some of the most efficient ways for detecting proximity to voltage collapse [10].

2 AC/DC System Model

In this section, a system model is built from different element models for generators, transmission network, loads, and the dc lines. This dynamic model corresponds to a typical representations of ac/dc power systems for voltage and transient stability analysis [11, 12, 13, 14, 15, 16, 17, 18]. Based on this model, it is shown below that a saddle-node bifurcation can be detected using power flow equations.

2.1 Generators

A synchronous generator model with constant terminal voltage and reactive power limits is used, to simulate the effects of a voltage regulator [15, 17]. Generator buses are numbered as $g = 1, 2, \ldots, n_G$, and generator n_G is the reference point for the bus voltage phasors.

$$
\begin{aligned}
\dot{\delta}_g &= \omega_g - \omega_{n_G} \qquad (1)\\
\dot{\omega}_g &= \frac{1}{M_g}\left(P_m - P_{gt} - D_g\omega_g\right)\\
P_{gt} &= \sum_{j=1}^{n}[G_{tj}V_tV_j\cos(\delta_t-\delta_j) + B_{tj}V_tV_j\sin(\delta_t-\delta_j)]\\
Q_{gt} &= \sum_{j=1}^{n}[G_{tj}V_tV_j\sin(\delta_t-\delta_j) - B_{tj}V_tV_j\cos(\delta_t-\delta_j)]
\end{aligned}
$$

Here, $G_{tj} + jB_{tj}$ is the tj term of the bus admittance matrix for an n bus system. $V_t\angle\delta_t$ is the voltage phasor at the generator terminal bus t, and P_{gt} and Q_{gt} are the active and reactive powers injected by the generator at its terminals. The reactive power Q_{gt} is assumed to be between limits Q_{max} and Q_{min}; when this variable exceeds these limits, Q_{gt} is fixed to the corresponding value and the terminal voltage magnitude V_t is allowed to change to meet this condition. M_g represents the machine inertia in (seconds)^2, and D_g is the generator's damping constant in seconds. The variable ω_g stands for the generator frequency deviation from the nominal frequency in rad/sec. The mechanical power applied to the generator shaft, P_m, is assumed constant at each load level.

2.2 Transmission System

A Π-equivalent circuit model is employed [15]. Transformers and phase shifters are included as part of the transmission network.

$$
\begin{aligned}
P_{sr} &= g_{sr}V_s^2 - g_{sr}V_sV_r\cos(\delta_s-\delta_r) + \qquad (2)\\
&\quad b_{sr}V_sV_r\sin(\delta_s-\delta_r)\\
Q_{sr} &= (b_s + b_{sr})V_s^2 - g_{sr}V_sV_r\sin(\delta_s-\delta_r) - \\
&\quad b_{sr}V_sV_r\cos(\delta_s-\delta_r)
\end{aligned}
$$

Here, P_{sr} and Q_{sr} are the transmitted powers between buses s and r. $V_r\angle\delta_r$ and $V_s\angle\delta_s$ are the voltage phasors at the respective buses. The transfer admittance is represented by $g_{sr} - jb_{sr}$, and $-jb_s$ denotes the shunt bus admittance at the end of the line/transformer.

2.3 Loads

Voltage and frequency dependent load models are used. These models are similar to the ones proposed in [11, 16, 17].

$$
\begin{aligned}
P_l &= -(P_{l_0} + \Delta P_{l_0}\lambda) - (P_{l_1} + \Delta P_{l_1}\lambda)(V_l/V_l^0)^2 - \qquad (3)\\
&\quad (P_{l_2} + \Delta P_{l_2}\lambda)(V_l/V_l^0) - D_{l_{fp}}(\dot{\delta}_l - \omega_{n_G}) - D_{l_{vp}}\dot{V}_l
\end{aligned}
$$

$$Q_l = -(Q_{l_0} + \Delta Q_{l_0}\lambda) - (Q_{l_1} + \Delta Q_{l_1}\lambda)(V_l/V_l^0)^2 - (Q_{l_2} + \Delta Q_{l_2}\lambda)(V_l/V_l^0) - D_{l_{fQ}}(\dot{\delta}_l - \omega_{n_G}) - D_{l_{vQ}}\dot{V}_l$$

where $l = 1 + n_G, 2 + n_G, \ldots, n_G + n_L$. Here P_l and Q_l are the powers injected by the load, and $V_l\angle\delta_l$ is the load phasor voltage at bus l. P_{l_0}, P_{l_1}, P_{l_2}, Q_{l_0}, Q_{l_1}, and Q_{l_2} are constant weighting factors that define the steady state base load. $D_{l_{fP}}$, $D_{l_{vP}}$, $D_{l_{fQ}}$, and $D_{l_{vQ}}$ represent the time constants of the frequency and voltage dependent dynamic terms in seconds. Note that these time constants can be set to zero to represent static load models. ΔP_{l_0}, ΔP_{l_1}, ΔP_{l_2}, ΔQ_{l_0}, ΔQ_{l_1}, ΔQ_{l_2}, and the parameter λ are used to simulate the slow time scale load change. For most studies of voltage collapse, it is assumed that the pattern of load change can be represented with one degree of freedom (λ) and that this evolution of load drives the system to a saddle-node bifurcation.

Notice that differential equations (3) can be rewritten as $\begin{bmatrix} k_1\dot{\delta}_l \\ k_2\dot{V}_l \end{bmatrix} = \begin{bmatrix} 1 & k_3 \\ 1 & k_4 \end{bmatrix} \begin{bmatrix} f_\lambda(\delta, \mathbf{V}) \\ g_\lambda(\delta, \mathbf{V}) \end{bmatrix}$, where $f_\lambda(\delta, \mathbf{V})$ and $g_\lambda(\delta, \mathbf{V})$ are the active and reactive power mismatches at the load buses, respectively.

2.4 HVDC Model

DC lines are simulated using R-L circuits, and the HVDC controllers are modeled using saturable PI current controllers. Although these control circuits are approximations to the more complicated HVDC control structures, they recreate several of the main properties of actual systems, especially when close to the equilibria. Voltage Dependent Current Order Limiter (VDCOL) can be introduced into this model by representing the controller current order as a nonlinear function of the ac converter voltages [12, 13]. This will be represented by assuming voltage dependence of the current order settings in the converter current controllers.

Assuming ideal harmonic filtering, equations (4) below can be used to simulate the behavior of the HVDC system under balanced 3-phase operation, including the control mode switching due to saturation of the converter current controllers [12, 13, 14, 15]. The convention used throughout this paper will apply the subscripts "r" and "i" to rectifier and inverter quantities, respectively.

$$\begin{aligned} \dot{I}_d &= (V_{d_r} - V_{d_i})/L_d - (R_d/L_d)I_d \\ \dot{x}_r &= \mathrm{l}_1(K_{I_r}[I_{o_r}(V_r) - I_d], y_r) \\ \dot{x}_i &= \mathrm{l}_1(K_{I_i}[I_d - I_{o_i}(V_i)], y_i) \end{aligned} \tag{4}$$

$$\begin{aligned} \cos\alpha_r &= \mathrm{l}_2(x_r + K_{P_r}[I_{o_r}(V_r) - I_d]) \\ V_{d_r} &= (3\sqrt{2}/\pi)a_r V_r \cos\alpha_r - (3/\pi)X_{c_r}I_d \\ S_r &= (V_n I_n/S_n)(3\sqrt{2}/\pi)a_r V_r I_d \\ P_r &= (V_n I_n/S_n)V_{d_r}I_d \\ Q_r &= \sqrt{S_r^2 - P_r^2} \end{aligned}$$

$$\begin{aligned} \cos\gamma_i &= \mathrm{l}_2(x_i + K_{P_i}[I_d - I_{o_i}(V_i)]) \\ V_{d_i} &= (3\sqrt{2}/\pi)a_i V_i \cos\gamma_i - (3/\pi)X_{c_i}I_d \\ S_i &= (V_n I_n/S_n)(3\sqrt{2}/\pi)a_i V_i I_d \\ P_i &= -(V_n I_n/S_n)V_{d_i}I_d \\ Q_i &= \sqrt{S_i^2 - P_i^2} \end{aligned}$$

Per unit norm[illegible]lization is emp[illegible]yed with I_n and V_n as the base qua[illegible]es f[illegible] dc [illegible]tem, [illegible] S_n [illegible] the base power for the ac side. V_{d_r} and V_{d_i} are the per unit dc terminal voltages at the rectifier and inverter ends. X_{c_r} and X_{c_i} are the per unit commutation reactance, and R_d and L_d are the per unit dc line parameters. The products $a_r V_r$ and $a_i V_i$ are the per unit ac bus voltages at the secondary ac side of the transformers. S_r and S_i are the magnitudes of the HVDC complex powers at the ac side, and P_r, P_i, Q_r, and Q_i are the active and reactive powers absorbed by the dc system. K_{I_r}, K_{I_i}, K_{P_r}, and K_{P_i} are the PI controller gains for each converter, respectively. The variables y_r and y_i are feedback measurements used to enforce firing and extinction angle limits in the current controllers to avert saturation problems during the numeric integration process; e.g., $y_r = x_r(t + \Delta t)$ and $y_i = x_i(t + \Delta t)$.

The controllers are designed so that their actions typically do not overlap, i.e., either the inverter or rectifier controls the dc current, but normally not both. For large dc transient currents produced by voltage changes at the converter buses, and under normal operating conditions the rectifier side controls the current while the inverter current controller is saturated at is minimum value γ_{min}. When the transient current is small the opposite applies. However, during recovery from abnormal operating conditions one can expect to have both converters controlling the current for small periods. Under fault conditions the current controllers can also be forced to their maximum limits $\alpha_{r_{max}}$ and $\gamma_{i_{max}}$. This behavior is approximated here by the limit functions $\mathrm{l}_1(\cdot)$ and $\mathrm{l}_2(\cdot)$, which are defined as follows:

$$\mathrm{l}_1(x, y) = \begin{cases} x & \text{if } y_{min} \le y \le y_{max} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{l}_2(x) = \begin{cases} x_{min} & \text{if } x \le x_{min} \\ x & \text{if } x_{min} < x < x_{max} \\ x_{max} & \text{if } x \ge x_{max} \end{cases}$$

At an equilibrium and in a region sufficiently close to it, the dc system operates in a a determined control mode, i.e., either the rectifier or the inverter current controllers control the current. Hence, the limit functions $\mathrm{l}_1(\cdot)$ and $\mathrm{l}_2(\cdot)$ are well defined at the equilibria, and so are the corresponding derivatives.

2.5 Vector Field Equations

The equations above can be rewritten in vector differential equation form to simplify the notation and analysis. Equations (1), (2), (3), and (1) can then be arranged into vector differential equations (5) below for an n bus ac/dc system ($n = n_G + n_L + 2n_{dc}$, where n_{dc} is the number of dc links in the system). The reference generator [illegible] is assumed to be an infinite bus so that all system equilibria are guaranteed to have $\omega_G = [\omega_1 \ldots \omega_{n_G-1}]$ $= 0$, otherwise the transfer conductance losses may produce a slight shift in generation frequency at the equilibria for the proposed system model. This condition is not necessary when all generator damping constants are set to zero.

$$\begin{aligned} \dot{\delta}_G &= \omega_G \\ \mathbf{M}_G\dot{\omega}_G &= \mathbf{f}_G(\delta, \mathbf{V}) - \mathbf{D}_G\omega_G \\ \mathbf{D_L}\begin{bmatrix} \dot{\delta}_L \\ \dot{\mathbf{V}}_L \end{bmatrix} &= \begin{bmatrix} \mathbf{f}_{L_\lambda}(\delta, \mathbf{V}) \\ \mathbf{g}_{L_\lambda}(\delta, \mathbf{V}) \end{bmatrix} \\ \dot{\mathbf{x}}_{dc} &= \mathbf{h}_{dc}(\mathbf{V}_{dc}, \mathbf{x}_{dc}, \mathbf{y}_{dc}) \\ 0 &= \mathbf{G}(\delta, \mathbf{V}, \mathbf{x}_{dc}, \mathbf{y}_{dc}) \end{aligned} \tag{5}$$

Vector $\mathbf{x}_{dc}$ stands for all the state variables for the system dc links operating at a fixed control mode, as defined by vector field $\mathbf{h}_{dc}(\cdot)$, so that the dc limit functions are well defined. On the other hand, $\mathbf{y}_{dc}$ represents the dc variables implicitly defined by the corresponding algebraic constraints in $\mathbf{G}(\cdot)$. $\mathbf{V}$ and δ depict all the voltage phasors at generator ($\mathbf{V}_G\angle\delta_G$), load ($\mathbf{V}_L\angle\delta_L$) and dc ($\mathbf{V}_{dc}\angle\delta_{dc}$) buses. Vector functions $\mathbf{f}_G(\cdot)$ and $\mathbf{f}_L(\cdot)$ stand for the active power mismatches at the generator and the load buses, respectively, and $\mathbf{g}_L(\cdot)$ corresponds to the reactive power mismatches at the load buses. Notice that the algebraic constraints represented by the vector field $\mathbf{G}(\cdot)$, correspond to the dc link constrains, the active and/or reactive power mismatches at dc and static load buses, and the reactive power equations at all generator buses. Finally, matrices $\mathbf{M}_G$, $\mathbf{D}_G$, and $\mathbf{D}_L$ are all nonsingular diagonal or block-diagonal matrices representing the generators' inertia and damping, and the loads' time constants, respectively.

Defining $\mathbf{x} \triangleq [\mathbf{x}_1^T\ \mathbf{x}_2]^T$ as some of the state variables associated to the system differential equations, and $\mathbf{y}$ as the system variables associated to the algebraic constrains, equations (5) can be rewritten as

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \omega_G \qquad (6)\\ \underbrace{\begin{bmatrix} \mathbf{M}_G & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}}_{\mathbf{K}} \begin{bmatrix} \dot{\omega}_G \\ \dot{\mathbf{x}}_2 \end{bmatrix} &= \underbrace{\begin{bmatrix} \mathbf{F}_{1_\lambda}(\mathbf{x},\mathbf{y}) - \mathbf{D}_G\omega_G \\ \mathbf{F}_{2_\lambda}(\mathbf{x},\mathbf{y}) \end{bmatrix}}_{\mathbf{F}_\lambda(\omega_G,\mathbf{x},\mathbf{y})} \\ \mathbf{0} &= \mathbf{G}_\lambda(\mathbf{x},\mathbf{y}) \end{aligned}$$

where $\mathbf{D}$ is a nonsingular matrix that contains the dynamic load time constant.

An equilibrium point for equations (6) can be obtained setting the left hand side to zero. Ordinary ac/dc power flow equations correspond to steady state equations $\mathbf{F}_\lambda(\mathbf{0},\mathbf{x},\mathbf{y}) = \mathbf{F}_\lambda(\mathbf{x},\mathbf{y}) = \mathbf{0}$ and $\mathbf{G}_\lambda(\mathbf{x},\mathbf{y}) = \mathbf{0}$, when the load is modeled as constant PQ. Different static load models could be implemented in the power flow equations, so that these equations can be used to directly determine equilibria of the dynamic model.

For the proposed dynamic system, nonsingularity of the Jacobian $D_y\mathbf{G}(\cdot)$ along system trajectories of interest guarantees a well posed system [19]. If matrix $D_y\mathbf{G}(\cdot)$ becomes singular, then the model represented by (6) breaks down, and singular perturbation techniques or dynamic load models should be used for this system representation. For example, a singular $D_y\mathbf{G}|_0$ at a particular equilibrium point $(\mathbf{0},\mathbf{x}_0,\mathbf{y}_0,\lambda_0)$ of a simple power system model consisting of one generator, a transmission line and a load, constitutes an impasse point that divides two stable equilibria in a system with no unstable equilibrium points; this type of behavior is non-physical in a real power network. The same phenomenon was observed in larger ac/dc systems, where two stable equilibria coalesce at a singular Jacobian. In this case, assuming a dynamic voltage dependence of some of the load buses removed the impasse point forcing one of the stable equilibria to become unstable, so that saddle-node theory can be applied to explain the bifurcation. When the algebraic constraints $\mathbf{G}(\cdot)$ have an invertible Jacobian $D_y\mathbf{G}(\cdot)$, variables $\mathbf{y} = \mathbf{h}_\lambda(\mathbf{y})$ can be eliminated (Implicit Function Theorem [20]), and equations (6) are reduced to

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \omega_G \qquad (7)\\ \mathbf{K}\begin{bmatrix} \dot{\omega}_G \\ \dot{\mathbf{x}}_2 \end{bmatrix} &= \mathbf{F}_\lambda(\omega_G,\mathbf{x},\mathbf{h}_\lambda(\mathbf{x})) \end{aligned}$$

which can be rewritten as $\mathbf{M}\dot{\mathbf{z}} = \mathbf{s}_\lambda(\mathbf{z})$, with $\mathbf{z} \triangleq [\mathbf{x}_1^T\ \omega_G^T\ \mathbf{x}_2^T]^T$, and $\mathbf{M} \triangleq \begin{bmatrix} \mathbf{I}_{n_G-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_G & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D} \end{bmatrix}$

3 Equivalency of Saddle-Node Bifurcation Conditions

A bifurcation, or structural instability, for $\mathbf{M}\dot{\mathbf{z}} = \mathbf{s}_\lambda(\mathbf{z})$ occurs when the Jacobian $D_z\mathbf{s}(\cdot)$ becomes singular at the equilibrium $(\mathbf{z}_0,\lambda_0)$. Several types of bifurcation are possible in this situation, but of these only the saddle-node occurs generically. Moreover, the following conditions apply at the saddle-node $(\mathbf{z}_0,\lambda_0)$ [6]:

1. $D_z\mathbf{s}|_0 \triangleq D_z\mathbf{s}_{\lambda_0}(\mathbf{z}_0)$ has a simple and unique zero eigenvalue, with normalized right eigenvector $\mathbf{v}$ and left eigenvector $\mathbf{w}$, i.e.,

$$D_z\mathbf{s}|_0\ \mathbf{v} = \mathbf{0} \quad \text{and} \quad \mathbf{w}^T\ D_z\mathbf{s}|_0 = \mathbf{0}^T \qquad (8)$$

2. $\mathbf{w}^T\ \left.\frac{\partial \mathbf{s}}{\partial\lambda}\right|_0 \neq 0.$ (9)

3. $\mathbf{w}^T\left[D_z^2\mathbf{s}|_0\ \mathbf{v}\right]\mathbf{v} \neq 0.$ (10)

Conditions 1 through 3 guarantee the generic quadratic behavior near the bifurcation point, and also prevent singular augmented Jacobians of the Newton- Raphson based methods used to determine bifurcation points as shown below.

Use of the vector field $\mathbf{s}_\lambda(\mathbf{z})$ in bifurcation analysis of power systems presents several problems, such as the difficulty of finding the explicit function $\mathbf{y} = \mathbf{h}_\lambda(\mathbf{z})$. Typically, function $\mathbf{h}(\cdot)$ cannot be expressed in closed form, though its derivatives are available. While ultimately only the derivatives of $\mathbf{h}(\cdot)$ are needed for computation of the saddle-node bifurcation point, use of the reduced equations sacrifices sparsity, significantly increasing the computational costs. Therefore, it is useful to relate known methods for detecting bifurcations based on saddle-node conditions for the reduced system (7) to conditions expressed in terms of the complete set of system equations (6).

An important computational issue in bifurcation studies for power systems is the relationship between eigenvalues of the Jacobian of the power flow equations[1], $\mathbf{J}_{PF}$, and those of the Jacobian of the system dynamic equations linearized at the equilibrium point, denoted by $\mathbf{J}_{TS} = \mathbf{M}^{-1}D_z\mathbf{s}|_0$. This subject has been studied for a number of ac only system models in [5, 8, 9]. Here it is shown that, for the proposed system model, the power flow equations can be used to directly detect a saddle-node bifurcation of system dynamic equations (7), under the generic assumption of "invertible" algebraic constraints $\mathbf{G}(\cdot)$; moreover, these equations yield information regarding right and left eigenvectors for the linearized equations of the reduce dynamic system (7). These ideas have been briefly discussed in [21], for a general dynamic ac system model with no algebraic constraints.

[1] The "power flow" equations in this paper differ slightly from the typical set of equations in the power system literature, since static load models used in the simulation are not only constant PQ models.

Thus, the power flow Jacobian at the equilibrium $(\mathbf{0}, \mathbf{x}_0, \mathbf{y}_0, \lambda_0)$ can be represented by

$$\mathbf{J}_{PF} = \left[\begin{array}{cc|c} D_{x_1}\mathbf{F}_1|_0 & D_{x_2}\mathbf{F}_1|_0 & D_y\mathbf{F}_1|_0 \\ D_{x_1}\mathbf{F}_2|_0 & D_{x_2}\mathbf{F}_2|_0 & D_y\mathbf{F}_2|_0 \\ \hline D_{x_1}\mathbf{G}|_0 & D_{x_2}\mathbf{G}|_0 & D_y\mathbf{G}|_0 \end{array}\right]$$

Hence, the determinant of $\mathbf{J}_{PF}$ can be calculated using

$$\begin{aligned} \det \mathbf{J}_{PF} &= \det(D_y\mathbf{G}|_0) \det\left(\begin{bmatrix} D_{x_1}\mathbf{F}_1|_0 & D_{x_2}\mathbf{F}_1|_0 \\ D_{x_1}\mathbf{F}_2|_0 & D_{x_2}\mathbf{F}_2|_0 \end{bmatrix} - \right. \\ &\qquad \left. \begin{bmatrix} D_y\mathbf{F}_1|_0 \\ D_y\mathbf{F}_2|_0 \end{bmatrix} D_y\mathbf{G}|_0^{-1} \begin{bmatrix} D_{x_1}\mathbf{G}|_0 & D_{x_2}\mathbf{G}|_0 \end{bmatrix}\right) \\ &= \det(D_y\mathbf{G}|_0)\det(\mathcal{P}) \end{aligned}$$

On the other hand, linearizing (7) at the equilibrium yields a block Jacobian structure satisfying

$$\mathbf{M}\,\mathbf{J}_{TS} = \qquad (11)$$

$$\begin{array}{c|ccc|} & \mathbf{x}_1 & \omega_G & \mathbf{x}_2 \\ \hline \dot{\mathbf{x}}_1 & \mathbf{0} & \mathbf{I}_{n_G-1} & \mathbf{0} \\ \hline \mathbf{M}_G\dot{\omega}_G & D_{x_1}\mathbf{F}_1|_0 + & -\mathbf{D}_G & D_{x_2}\mathbf{F}_1|_0 + \\ & D_y\mathbf{F}_1|_0 D_{x_1}\mathbf{h}|_0 & & D_y\mathbf{F}_1|_0 D_{x_2}\mathbf{h}|_0 \\ \mathbf{D}\dot{\mathbf{x}}_2 & D_{x_1}\mathbf{F}_2|_0 + & \mathbf{0} & D_{x_2}\mathbf{F}_2|_0 + \\ & D_y\mathbf{F}_2|_0 D_{x_1}\mathbf{h}|_0 & & D_y\mathbf{F}_2|_0 D_{x_2}\mathbf{h}|_0 \\ \hline \end{array}$$

Thus, using standard block determinant formulas and since $D_x\mathbf{h}|_0 = -D_y\mathbf{G}|_0^{-1}D_x\mathbf{G}|_0$, $\det(D_z\mathbf{s}|_0) = (-1)^k \det \mathbf{J}_{PF}/\det(D_y\mathbf{G}|_0)$, where k is a positive integer. This equation shows that a singular linearized dynamic equations Jacobian implies having a singular power flow Jacobian, since $\mathbf{G}(\cdot)$ is bounded above. Moreover, theorems 1 and 2 below prove that all saddle-node bifurcation conditions, represented by (8), (9) and (10), have an analogous representation in the power flow equations. In order to simplify the notation, the vector fields $\mathbf{F}(\cdot)$ and $\mathbf{G}(\cdot)$, which represent the power flow equations of the proposed ac/dc system model, are grouped in the vector field $\mathcal{F}(\cdot) \triangleq [\mathbf{F}^T(\cdot)\ \mathbf{G}^T(\cdot)]^T$.

Theorem 1 *Let $D_y\mathbf{G}|_0$ be nonsingular at the saddle-node bifurcation point $(\mathbf{0}, \mathbf{x}_0, \mathbf{y}_0, \lambda_0) \in \mathbf{R}^{N+1}$ ($N = 3n_G + 2n_L + 9n_{dc} - 3$) satisfying transversality conditions (8), (9) and (10). Let $\mathcal{F} : \mathbf{R}^{N-n_G+1} \times \mathbf{R} \mapsto \mathbf{R}^{N-n_G+1}$ be as defined above. Hence, for properly normalized left eigenvectors $\mathbf{w} \in \mathbf{R}^m$ (m = number of state variables in (7)) and $\varpi \in \mathbf{R}^{N-n_G+1}$, corresponding to zero eigenvalues of $D_z\mathbf{s}|_0$ and $D_{(x,y)}\mathcal{F}|_0$, respectively, it follows that*

$$\mathbf{w}^T \left.\frac{\partial \mathbf{s}}{\partial\lambda}\right|_0 = \varpi^T \left.\frac{\partial \mathcal{F}}{\partial\lambda}\right|_0 \qquad (12)$$

Furthermore, uniqueness and simplicity of the zero eigenvalue of $D_z\mathbf{s}|_0$ guarantees having a unique and simple zero eigenvalue for matrix $D_{(x,y)}\mathcal{F}|_0$.

Proof: Since $D_y\mathbf{G}|_0$ is nonsingular at the equilibrium, there exists a smooth local function $\mathbf{h}(\cdot)$ around $(\mathbf{0}, \mathbf{x}_0, \mathbf{y}_0, \lambda_0)$ such that $\mathbf{y} = \mathbf{h}_\lambda(\mathbf{x})$ (Implicit Function Theorem [20]), and $\partial\mathbf{h}/\partial\lambda|_0 = -D_y\mathbf{G}|_0^{-1}\partial\mathbf{G}/\partial\lambda|_0$. On the other hand, based on the definitions of $\mathbf{s}_\lambda(\mathbf{z})$ and $\mathbf{w} = [\mathbf{w}_{x_1}^T\ \mathbf{w}_{\omega_G}^T\ \mathbf{w}_{x_2}^T]^T$, then $\mathbf{w}^T D_z\mathbf{s}|_0 = \mathbf{0}^T$ implies from (11) that

$$\mathbf{w}_{[illegible]} = \mathbf{0}\ [illegible]\ \mathbf{w}_{\omega_G} \qquad (13)$$
$$\mathbf{w}_{\omega_G}^T\ [illegible]\ D_x\mathbf{F}\ [illegible]\ D_y\mathbf{G}\ [illegible]\ \mathbf{G}|_0$$

Now, it follows from $[\varpi_x^T\ \varpi_y^T]\ D_{(x,y)}\mathcal{F}|_0 = \mathbf{0}^T$ that

$$\begin{aligned} \varpi_y^T &= -\varpi_x^T D_y\mathbf{F}|_0 D_y\mathbf{G}|_0^{-1} \\ \varpi_x^T(D_x\mathbf{F}|_0 &- D_y\mathbf{F}|_0 D_y\mathbf{G}|_0^{-1}D_x\mathbf{G}|_0) = \mathbf{0}^T \end{aligned} \qquad (14)$$

For properly scaled eigenvectors, from (13) and (14), and since $\mathbf{w}$ is unique by definition, one has a unique eigenvector $\varpi = [\varpi_x^T\ \varpi_y^T]^T$, with $\varpi_x = [\mathbf{w}_{\omega_G}^T\ \mathbf{w}_{x_2}^T]^T$. Hence, $D_{(x,y)}\mathcal{F}|_0$ has a unique and simple zero eigenvalue at the bifurcation point. Furthermore,

$$\begin{aligned} \mathbf{w}^T \left.\frac{\partial \mathbf{s}}{\partial\lambda}\right|_0 &= [\mathbf{w}_{\omega_G}^T\ \mathbf{w}_{x_2}^T]\left(\left.\frac{\partial \mathbf{F}}{\partial\lambda}\right|_0 - D_y\mathbf{F}|_0 D_y\mathbf{G}|_0^{-1}\left.\frac{\partial \mathbf{G}}{\partial\lambda}\right|_0\right) \\ &= \varpi_x^T\left.\frac{\partial \mathbf{F}}{\partial\lambda}\right|_0 - \varpi_x^T D_y\mathbf{F}|_0 D_y\mathbf{G}|_0^{-1}\left.\frac{\partial \mathbf{G}}{\partial\lambda}\right|_0 \\ &= \varpi^T\left.\frac{\partial \mathcal{F}}{\partial\lambda}\right|_0 \end{aligned}$$

□

Theorem 2 *Let $D_y\hat{\mathbf{G}}|_0$ be nonsingular at the saddle-node bifurcation point $(\mathbf{0}, \mathbf{x}_0, \mathbf{y}_0, \lambda_0) \in \mathbf{R}^{N+1}$ satisfying transversality conditions (8), (9) and (10). Let $\mathcal{F}(\cdot)$, $\mathbf{w}$, and ϖ be defined as in theorem 1. Then, for properly normalized right eigenvectors $\mathbf{v} \in \mathbf{R}^m$ and $v \in \mathbf{R}^{N-n_G+1}$, corresponding to simple and unique zero eigenvalues of $D_z\mathbf{s}|_0$ and $D_{(x,y)}\mathcal{F}|_0$, respectively, i.e., $D_z\mathbf{s}|_0\mathbf{v} = \mathbf{0}$, and $D_{(x,y)}\mathcal{F}|_0 v = \mathbf{0}$, it follows that*

$$\mathbf{w}^T\left[D_z^2\mathbf{s}|_0\mathbf{v}\right]\mathbf{v} = \varpi^T\left[D_{(x,y)}^2\mathcal{F}|_0 v\right] v \qquad (15)$$

Proof: Following similar arguments to the ones employed in the proof of theorem 1, one has that for $\mathbf{v} = [\mathbf{v}_{x_1}^T\ \mathbf{v}_{\omega_G}^T\ \mathbf{v}_{x_2}^T]^T$ and $v = [v_x^T\ v_y^T]^T$, $\mathbf{v}_{\omega_G} = \mathbf{0}$, $v_y = -D_y\mathbf{G}|_0^{-1}D_x\mathbf{G}|_0 v_x$, and $v_x = [\mathbf{v}_{x_1}^T\ \mathbf{v}_{x_2}^T]^T \triangleq \mathbf{v}_x$. Furthermore, the product of the tensor $D_{(x,y)}^2\mathcal{F}|_0$ and the vector v, yields the matrix

$$\begin{aligned} D_{(x,y)}^2\mathcal{F}|_0 v &= \\ &\begin{bmatrix} D_x^2\mathbf{F}|_0 v_x + D_{xy}^2\mathbf{F}|_0 v_y & D_{yx}^2\mathbf{F}|_0 v_x + D_y^2\mathbf{F}|_0 v_y \\ D_x^2\mathbf{G}|_0 v_x + D_{xy}^2\mathbf{G}|_0 v_y & D_{yx}^2\mathbf{G}|_0 v_x + D_y^2\mathbf{G}|_0 v_y \end{bmatrix} \\ = &\left[\begin{matrix} D_x^2\mathbf{F}|_0 v_x - D_{xy}^2\mathbf{F}|_0 D_y\mathbf{G}|_0^{-1}D_x\mathbf{G}|_0 v_x \\ D_x^2\mathbf{G}|_0 v_x - D_{xy}^2\mathbf{G}|_0 D_y\mathbf{G}|_0^{-1}D_x\mathbf{G}|_0 v_x \end{matrix}\right. \\ &\left.\begin{matrix} D_{yx}^2\mathbf{F}|_0 v_x - D_y^2\mathbf{F}|_0 D_y\mathbf{G}|_0^{-1}D_x\mathbf{G}|_0 v_x \\ D_{yx}^2\mathbf{G}|_0 v_x - D_y^2\mathbf{G}|_0 D_y\mathbf{G}|_0^{-1}D_x\mathbf{G}|_0 v_x \end{matrix}\right] \end{aligned}$$

Hence, it can be shown that

$$\begin{aligned} \varpi^T[D_{(x,y)}^2\mathcal{F}|_0 v]v &= \qquad (16) \\ &\left[\ \varpi_x^T \ \middle|\ -\varpi_x^T D_y\mathbf{F}|_0 D_y\mathbf{G}|_0^{-1}\ \right][D_{(x,y)}^2\mathcal{F}|_0 v] \\ &\left[\frac{v_x}{-D_y\mathbf{G}|_0^{-1}D_x\mathbf{G}|_0 v_x}\right] \\ = &\ \varpi_x^T\Big[D_x^2\mathbf{F}|_0 v_x - D_{xy}^2\mathbf{F}|_0 D_y\mathbf{G}|_0^{-1}D_x\mathbf{G}|_0 v_x - \\ &D_{yx}^2\mathbf{F}|_0 v_x D_y\mathbf{G}|_0^{-1}D_x\mathbf{G}|_0 + D_y^2\mathbf{F}|_0 D_y\mathbf{G}|_0^{-1}D_x\mathbf{G}|_0 v_x \\ &D_y\mathbf{G}|_0^{-1}D_x\mathbf{G}|_0 - D_y\mathbf{F}|_0 D_y\mathbf{G}|_0^{-1}D_x^2\mathbf{G}|_0 v_x + \\ &D_y\mathbf{F}|_0 D_y\mathbf{G}|_0^{-1}D_{xy}^2\mathbf{G}|_0 D_y\mathbf{G}|_0^{-1}D_x\mathbf{G}|_0 v_x + \\ &D_y\mathbf{F}|_0 D_y\mathbf{G}|_0^{-1}D_{yx}^2\mathbf{G}|_0 v_x D_y\mathbf{G}|_0^{-1}D_x\mathbf{G}|_0 - \\ &D_y\mathbf{F}|_0 D_y\mathbf{G}|_0^{-1}D_y^2\mathbf{G}|_0 D_y\mathbf{G}|_0^{-1}D_x\mathbf{G}|_0 v_x \\ &D_y\mathbf{G}|_0^{-1}D_x\mathbf{G}|_0\Big]v_x \end{aligned}$$

On the other hand, since in a neighborhood of the bifurcation point the Implicit Function theorem implies that $\mathbf{y} = \mathbf{h}_\lambda(\mathbf{x})$, it follows that

$$[illegible]\ \mathbf{s}_\lambda(\mathbf{z})\mathbf{v}\ [illegible] \begin{bmatrix} \mathbf{0} \\ D_x\mathbf{F}_\lambda(\ [illegible]\ \mathbf{x}, \mathbf{h}_\lambda(\mathbf{x}))\mathbf{v}_x + \\ [illegible]\ \mathbf{F}_\lambda(\omega_G,\ [illegible]\ \mathbf{h}_\lambda(\mathbf{x}))\ [illegible]\ (\mathbf{x})\ [illegible] \end{bmatrix}$$

Then

$$[D_z^2 \mathbf{s}|_0 \mathbf{v}]\mathbf{v} = \left[\begin{array}{c} \mathbf{0} \\ (D_x^2 \mathbf{F}|_0 \mathbf{v}_x + D_{yx}^2 \mathbf{F}|_0 \mathbf{v}_x \, D_x \mathbf{h}|_0 + D_{xy}^2 \mathbf{F}|_0 D_x \mathbf{h}|_0 \mathbf{v}_x + \\ D_y^2 \mathbf{F}|_0 D_x \mathbf{h}|_0 \mathbf{v}_x \, D_x \mathbf{h}|_0 + D_y \mathbf{F}|_0 D_x \mathbf{p}|_0) \mathbf{v}_x \end{array} \right] \quad (17)$$

where $\mathbf{p}_\lambda(\mathbf{x}) \triangleq D_x \mathbf{h}_\lambda(\mathbf{x}) \mathbf{v}_x$. Now, since $\mathbf{0} = D_x \mathbf{G}_\lambda(\mathbf{x}, \mathbf{h}_\lambda(\mathbf{x})) + D_y \mathbf{G}_\lambda(\mathbf{x}, \mathbf{h}_\lambda(\mathbf{x})) D_x \mathbf{h}_\lambda(\mathbf{x})$ and, hence, $D_x \mathbf{h}|_0 = -D_y \mathbf{G}|_0^{-1} D_x \mathbf{G}|_0$, it can be shown that

$$\begin{aligned} \mathbf{0} \;=\; & D_x^2 \mathbf{G}_\lambda(\mathbf{x}, \mathbf{h}_\lambda(\mathbf{x}))\mathbf{v}_x + D_{yx}^2 \mathbf{G}_\lambda(\mathbf{x}, \mathbf{h}_\lambda(\mathbf{x}))\mathbf{v}_x D_x \mathbf{h}_\lambda(\mathbf{x}) + \\ & D_{xy}^2 \mathbf{G}_\lambda(\mathbf{x}, \mathbf{h}_\lambda(\mathbf{x}))\mathbf{p}_\lambda(\mathbf{x}) + D_y^2 \mathbf{G}_\lambda(\mathbf{x}, \mathbf{h}_\lambda(\mathbf{x}))\mathbf{p}_\lambda(\mathbf{x}) \\ & D_x \mathbf{h}_\lambda(\mathbf{x}) + D_y \mathbf{G}_\lambda(\mathbf{x}, \mathbf{h}_\lambda(\mathbf{x})) D_x \mathbf{p}_\lambda(\mathbf{x}) \end{aligned}$$

$$\begin{aligned} \Rightarrow \; D_x \mathbf{p}|_0 = \; & \\ & -D_y \mathbf{G}|_0^{-1} \left[D_x^2 \mathbf{G}|_0 \mathbf{v}_x + D_{yx}^2 \mathbf{G}|_0 \mathbf{v}_x \, D_x \mathbf{h}|_0 + \right. \\ & \left. D_{xy}^2 \mathbf{G}|_0 D_x \mathbf{h}|_0 \mathbf{v}_x + D_y^2 \mathbf{G}|_0 D_x \mathbf{h}|_0 \mathbf{v}_x \, D_x \mathbf{h}|_0 \right] \end{aligned} \quad (18)$$

And from equations (17) and (18), it follows that

$$\begin{aligned} \mathbf{w}[D_z^2 \mathbf{s}|_0 \mathbf{v}]\mathbf{v} = \; & \\ & [\mathbf{w}_{\omega_G}^T \; \mathbf{w}_{x_2}^T] \Big[D_x^2 \mathbf{F}|_0 \mathbf{v}_x - D_{xy}^2 \mathbf{F}|_0 D_y \mathbf{G}|_0^{-1} D_x \mathbf{G}|_0 \mathbf{v}_x - \\ & D_{yx}^2 \mathbf{F}|_0 \mathbf{v}_x D_y \mathbf{G}|_0^{-1} D_x \mathbf{G}|_0 + D_y^2 \mathbf{F}|_0 D_y \mathbf{G}|_0^{-1} D_x \mathbf{G}|_0 \mathbf{v}_x \\ & D_y \mathbf{G}|_0^{-1} D_x \mathbf{G}|_0 - D_y \mathbf{F}|_0 D_y \mathbf{G}|_0^{-1} D_x^2 \mathbf{G}|_0 \mathbf{v}_x + \\ & D_y \mathbf{F}|_0 D_y \mathbf{G}|_0^{-1} D_{xy}^2 \mathbf{G}|_0 D_y \mathbf{G}|_0^{-1} D_x \mathbf{G}|_0 \mathbf{v}_x + \\ & D_y \mathbf{F}|_0 D_y \mathbf{G}|_0^{-1} D_{yx}^2 \mathbf{G}|_0 \mathbf{v}_x D_y \mathbf{G}|_0^{-1} D_x \mathbf{G}|_0 - \\ & D_y \mathbf{F}|_0 D_y \mathbf{G}|_0^{-1} D_y^2 \mathbf{G}|_0 D_y \mathbf{G}|_0^{-1} D_x \mathbf{G}|_0 \mathbf{v}_x \\ & D_y \mathbf{G}|_0^{-1} D_x \mathbf{G}|_0 \Big] \mathbf{v}_x \end{aligned}$$

Therefore, from (16),

$$\mathbf{w}^T \left[D_z^2 \mathbf{s}|_0 \mathbf{v} \right] \mathbf{v} = \varpi^T \left[D_{(x,y)}^2 \mathcal{F}|_0 v \right] v$$

□

4 Nonsingularity Conditions

There are several well known techniques for detecting bifurcations in dynamic systems [7]. However, in power system analysis two methods, namely, direct (Point of Collapse or PoC) and continuation methods, have been used to detect saddle- node bifurcations in a variety of network models [22, 23, 24, 25, 26, 27, 28, 29]. One of the main concerns when applying these methods is the singularity of the corresponding equations' Jacobians at the bifurcation point, which could cause numerical problems when a Newton-Raphson solution algorithm is used. In this section, the formal proofs of nonsingularity of the Jacobians at a saddle-node bifurcation point for both methods are presented.

4.1 Direct Methods

Using $\mathcal{F}(\cdot)$ and ϖ as defined above, and defining $\chi \triangleq [\mathbf{x}^T \; \mathbf{y}^T]^T$, the left eigenvector PoC equations can be written as

$$\begin{aligned} D_\chi \mathcal{F}_\lambda(\chi)^T \varpi &= \mathbf{0} \\ \mathcal{F}_\lambda(\chi) &= \mathbf{0} \\ \frac{\partial \mathcal{F}_\lambda}{\partial \lambda}^T \varpi &= k \end{aligned} \quad (19)$$

Where k is any scalar different from zero. Hence, the Jacobian for (19) at the bifurcation point is

$$\mathbf{J}_{PoC} = \left[\begin{array}{ccc} D_\chi^2 \mathcal{F}|_0^T \varpi & D_\chi \mathcal{F}|_0^T & D_{\chi\lambda} \mathcal{F}|_0^T \varpi \\ D_\chi \mathcal{F}|_0 & \mathbf{0} & \left.\frac{\partial \mathcal{F}}{\partial \lambda}\right|_0 \\ \varpi^T D_{\chi\lambda} \mathcal{F}|_0 & \left.\frac{\partial \mathcal{F}}{\partial \lambda}\right|_0^T & \varpi^T \left.\frac{\partial^2 \mathcal{F}}{\partial \lambda^2}\right|_0 \end{array} \right] \quad (20)$$

In spite of the individual block $D_\chi \mathcal{F}|_0$ being singular, conditions (9) and (10) guarantee that $\mathbf{J}_{PoC}$ is nonsingular, as shown in theorem 3 [30].

Theorem 3 *Let $\mathbf{s}(\cdot)$, $\mathcal{F}(\cdot)$, χ, v, ϖ, and λ be defined as shown previously. Then, if vector field $\mathbf{s}(\cdot)$ has a saddle-node bifurcation satisfying transversality conditions (8), (9) and (10) at the equilibrium $(\mathbf{0}, \chi_0, \lambda_0)$, and matrix $D_y \mathbf{G}|_0$ is nonsingular, then matrix $\mathbf{J}_{PoC}$ is also nonsingular.*

Proof: In order to prove $\mathbf{J}_{PoC}$ is nonsingular, one has to prove that the only vector that satisfies equation $\mathbf{J}_{PoC}[\mathbf{p}^T \; \mathbf{q}^T \; r]^T = \mathbf{0}$ is the zero vector, i.e., $\mathbf{p} = \mathbf{q} = \mathbf{0} \in \mathbf{R}^{N-n_G+1}$, and $r = 0 \in \mathbf{R}$. Thus, from (20) one has that

$$\begin{aligned} & D_\chi \mathcal{F}|_0 \mathbf{p} + \left.\frac{\partial \mathcal{F}}{\partial \lambda}\right|_0 r = \mathbf{0} \\ & \Rightarrow \rho^T \left.\frac{\partial \mathcal{F}}{\partial \lambda}\right|_0 r = -\rho^T D_\chi \mathcal{F}|_0 \mathbf{p} \quad \forall \rho \in \mathbf{R}^{N-n_G+1} \end{aligned}$$

In particular, if $\rho = \varpi$, then $\varpi^T \partial \mathcal{F}/\partial\lambda|_0 r = -\varpi^T D_\chi \mathcal{F}|_0 \mathbf{p} = 0$. Hence, from (9) and (12) it follows that $r = 0$, and $\mathbf{p} = kv$, $k \neq 0 \in \mathbf{R}$, or $\mathbf{p} = \mathbf{0}$. If $\mathbf{p} = kv$, then from (20)

$$\begin{aligned} & k[D_\chi^2 \mathcal{F}|_0^T \varpi]v + D_\chi \mathcal{F}|_0^T \mathbf{q} = \mathbf{0} \\ & \Rightarrow k\varpi^T [D_\chi^2 \mathcal{F}|_0 v]\rho = -\mathbf{q}^T D_\chi \mathcal{F}|_0 \rho \quad \forall \rho \in \mathbf{R}^{N-n_G+1} \end{aligned}$$

If $\rho = v$, then $k\varpi^T [D_\chi^2 \mathcal{F}|_0 v]v = -\mathbf{q}^T D_\chi \mathcal{F}|_0 v = 0$. But from (10) and (15), this is a contradiction; therefore, $\mathbf{p} = \mathbf{0}$ and $D_\chi \mathcal{F}|_0^T \mathbf{q} = \mathbf{0}$. Hence, $\mathbf{q} = k\varpi$, $k \neq 0 \in \mathbf{R}$, or $\mathbf{q} = \mathbf{0}$. If $\mathbf{q} = k\varpi$, from (20) it follows that $k \partial \mathcal{F}/\partial\lambda|_0^T \varpi = 0$, which is also a contradiction. Hence, $\mathbf{q} = \mathbf{p} = \mathbf{0}$ and $r = 0$. □

The same arguments can be used to formally prove nonsingularity of the Jacobians corresponding to the alternate PoC equations (21) shown below. These equations have proven to be computationally more efficient than (19) [27].

$$\begin{array}{ll} D_\chi \mathcal{F}_\lambda(\chi) v = \mathbf{0} & D_\chi \mathcal{F}_\lambda(\chi)^T \varpi = \mathbf{0} \\ \mathcal{F}_\lambda(\chi) = \mathbf{0} & \mathcal{F}_\lambda(\chi) = \mathbf{0} \\ \| v \|_\infty = 1 & \| \varpi \|_\infty = 1 \end{array} \quad (21)$$

4.2 Continuation Methods

Continuation methods have been used since the 60's in a variety of engineering fields [7]. The method presented here uses parameterization and perpendicular intersection techniques, to trace the branch of equilibria associated to saddle-node bifurcations in ac/dc networks [27, 28, 29].

The parameterized continuation method consists of a three step approach to tracing the equilibrium points as one parameter in the system changes, i.e., find the solutions to the steady state equations $\mathbf{s}_\lambda(\mathbf{z}) = \mathbf{0}$, for a given set of parameter values. Typically, the loading factor λ is the varying parameter; however, as the system gets closer to bifurcation, the Jacobian $D_z \mathbf{s}|_0$ becomes ill- conditioned. Thus, a change of parameter, such as switching from λ to a state variable $z_i \in \mathbf{z}$, makes the Jacobian nonsingular.

For the power system dynamic model used throughout this paper, where the power flow equations are represented by the vector field $\mathcal{F}(\cdot)$, and for a nonsingular Jacobian $D_y \mathbf{G}|_0$, there is a one to one correspondence between the equilibria of $\mathbf{s}(\cdot)$ and the solutions of

$_\lambda(\chi) = \mathbf{0}$. Furthermore, based on theorems 1 and 2, a saddle-node bifurcation in the vector field $\mathbf{s}(\cdot)$ can be detected using $\mathcal{F}(\cdot)$. Therefore, one can trace a bifurcation branch utilizing the power flow equations.

The continuation method algorithm can then be summarized in the following three steps:

i) *Predictor*: From $\mathcal{F}_\lambda(\chi) = \mathcal{F}_p(\hat{\chi}) = \mathbf{0}$, where initially $\hat{\chi} = \chi$ and $p = \lambda$, it follows that

$$D_{\hat{\chi}}\mathcal{F}(\hat{\chi}_1, p_1)\frac{d\hat{\chi}}{dp} = -\left.\frac{\partial \mathcal{F}}{\partial p}\right|_1 \qquad (22)$$

where p_1 and $\hat{\chi}_1$ come from a previous iteration. Hence, the direction $\Delta\hat{\chi}$ to move in state space can be found by solving equation (22), so that $\Delta\hat{\chi} \triangleq \Delta p\, d\hat{\chi}/dp$, where the parameter increment Δp can be defined as a function of a scaling constant k to vary the speed at which the equilibrium branch is traced, i.e., $\Delta p \triangleq k \parallel d\hat{\chi}/dp \parallel^{-1}$. As the process approaches the bifurcation, p is likely to change to one of the ac bus voltages, according to step iii, with the loading factor λ becoming part of $\hat{\chi}$.

ii) *Corrector*: Find the intersection between the perpendicular hyperplane to the tangent vector and the equilibrium branch, i.e., solve equations

$$\begin{aligned} \mathcal{F}(\hat{\chi}, p) &= \mathbf{0} \qquad (23)\\ \Delta p\,(p - p_1 - \Delta p) + \Delta\hat{\chi}^T(\hat{\chi} - \hat{\chi}_1 - \Delta\hat{\chi}) &= 0 \end{aligned}$$

iii) *Parameterization*: Check the relative change in all the system variables, and trade p with the variable that presents the largest change. In other words,

$$p \leftarrow \max\left\{\left|\frac{\Delta\hat{\chi}_1}{\hat{\chi}_1}\right|, \left|\frac{\Delta\hat{\chi}_2}{\hat{\chi}_2}\right|, \ldots, \left|\frac{\Delta\hat{\chi}_n}{\hat{\chi}_n}\right|, \left|\frac{\Delta p}{p}\right|\right\}$$

By changing the parameter p from λ to a state variable $\chi_i \in \chi$, one guarantees that the Jacobian of equations (22) is nonsingular at the bifurcation point. Furthermore, using techniques similar to the ones used in the proof of theorem 3, it can be shown that the Jacobian of equations (23) is also nonsingular at the bifurcation point, even for $p = \lambda$ (singular power flow Jacobian $D_\chi\mathcal{F}|_0$).

Theorem 4 *Let $\mathcal{F}(\cdot)$, χ, and λ be defined as in theorem 3. At the bifurcation point $(\mathbf{0}, \chi_0, \lambda_0)$, satisfying transversality conditions (8), (9) and (10), let the parameter $p_0 \in \mathbf{R}$ and the vector $\hat{\chi}_0 \in \mathbf{R}^{N-n_G+1}$ be defined as $p_0 \triangleq \chi_{0_M}$, for $|v_M| \geq |v_i|\ \forall\ i \neq M$, $v = [v_1 \cdots v_{N-n_G+1}]^T$, and $\hat{\chi}_0 \triangleq [\chi_{0_1} \cdots \chi_{0_{M-1}}\ \lambda_0\ \chi_{0_{M+1}} \cdots \chi_{0_{N-n_G+1}}]^T$. Then, $\mathcal{F}_{p_0}(\hat{\chi}_0) = \mathbf{0}$, and $D_{\hat{\chi}}\mathcal{F}|_0$ is nonsingular if vectors $\partial\mathcal{F}/\partial\lambda|_0 = \partial\mathcal{F}/\partial\hat{\chi}_M|_0$, and $\partial\mathcal{F}/\partial\chi_M|_0 = \partial\mathcal{F}/\partial p|_0$ in $\mathbf{R}^{N-n_G+1}$ are not collinear.*

Proof: From theorem 1, matrix $D_\chi\mathcal{F}|_0$ has a simple and unique zero eigenvalue at the saddle-node bifurcation $(\mathbf{0}, \chi_0, \lambda_0)$, with a unique right eigenvector v. Then, equation $D_\chi\mathcal{F}|_0 v = \mathbf{0}$ can be rewritten as

$$\sum_{j=1}^{N-n_G+1} v_j \left.\frac{\partial\mathcal{F}}{\partial\chi_j}\right|_0 = \mathbf{0} \quad\Rightarrow\quad \left.\frac{\partial\mathcal{F}}{\partial\chi_M}\right|_0 = \sum_{i=1, i\neq M}^{N-n_G+1} \alpha_i \left.\frac{\partial\mathcal{F}}{\partial\chi_i}\right|_0$$

where all $\alpha_i \triangleq v_i/v_M$ are unique, and $v_M \neq 0$ by definition. Therefore, replacing column M in $D_\chi\mathcal{F}|_0$ by vector $\partial\mathcal{F}/\partial\lambda|_0$ makes matrix $D_{\hat{\chi}}\mathcal{F}|_0$ nonsingular at the saddle-node bifurcation point, since this vector is not collinear with $\partial\mathcal{F}/\partial\chi_M|_0$ by assumption. □

The assumption of lack of collinearity between vectors $\partial\mathcal{F}/\partial\lambda|_0$ and $\partial\mathcal{F}/\partial y_M|_0$ is generic, since it is likely that for some index j, $1 \leq j \leq N - n_G + 1$, $\partial\mathcal{F}_j/\partial\lambda|_0 \neq 0$ while $\partial\mathcal{F}_j/\partial y_M|_0 = 0$.

At the saddle-node bifurcation $(\mathbf{0}, \chi_0, \lambda_0)$ satisfying the transversality conditions (8), (9) and (10), predictor equations (22) can be rewritten, for $\hat{\chi} = \chi$ and $p = \lambda$, as $D_\chi\mathcal{F}|_0 t_\chi = -\partial\mathcal{F}/\partial\lambda|_0 t_\lambda$, where $[t_\chi^T\ t_\lambda]^T$ is the tangent vector to the manifold $\mathcal{F}_\lambda(\chi) = 0$ in $\mathbf{R}^{N-n_G+1} \times \mathbf{R}$ at the bifurcation point. Then, from transversality condition (12) it follows that

$$\begin{aligned} \varpi^T D_\chi\mathcal{F}|_0 t_\chi &= -\varpi^T \left.\frac{\partial\mathcal{F}}{\partial\lambda}\right|_0 t_\lambda \\ \Rightarrow D_\chi\mathcal{F}|_0 t_\chi &= c\, D_\chi\mathcal{F}|_0 v = \mathbf{0} \qquad c \neq 0 \in \mathbf{R} \end{aligned}$$

Therefore, at the bifurcation, the tangent vector of the corrector step is collinear with the right eigenvector v. This implies, from theorem 4, that a proper parameterization should make the Jacobian of predictor equations (22) generically nonsingular along the bifurcation branch, since singularity points represented by saddle-nodes can be parameterized to avoid numerical problems.

Theorem 5 *Let $\mathcal{F}(\cdot)$, χ and λ be defined as in theorem 3. Let the bifurcation branch of equilibria be represented by the smooth manifold $\mathcal{F}_\lambda(\chi) = \mathbf{0}$ in $\mathbf{R}^{N-n_G+1} \times \mathbf{R}$ (limits are not considered as in previous theorems), in a neighborhood of the saddle-node bifurcation $(\mathbf{0}, \chi_0, \lambda_0)$ satisfying transversality conditions (8), (9), and (10). Furthermore, let the tangent vector $\mathbf{t}_\chi = d\chi/d\lambda$ be defined by the predictor equation $D_\chi\mathcal{F}_{\lambda_*}(\chi_*)\,\mathbf{t}_\chi = -\partial\mathcal{F}/\partial\lambda|_*$, where $\mathcal{F}_{\lambda_*}(\chi_*) = \mathbf{0}$. Assume that $\mathbf{t}_\chi$ is nonperpendicular to the properly normalized unique right eigenvector v defined in theorem 2. Then, the Jacobian of the corrector equations (23), i.e.,*

$$\mathbf{J}_C = \begin{bmatrix} D_\chi\mathcal{F}|_0 & \left.\frac{\partial\mathcal{F}}{\partial\lambda}\right|_0 \\ \Delta\chi^T & \Delta\lambda \end{bmatrix} \qquad (24)$$

is nonsingular at the saddle-node bifurcation point. Here, $\Delta\lambda \triangleq k \parallel \mathbf{t}_\chi \parallel^{-1}$, $\Delta\chi \triangleq \Delta\lambda\,\mathbf{t}_\chi$, and the scalar $k \neq 0 \in \mathbf{R}$ is chosen so that the saddle-node bifurcation $(\mathbf{0}, \chi_0, \lambda_0)$ is the solution to predictor equations (23).

Proof: Once again, to prove that $\mathbf{J}_C$ is nonsingular, one has to prove that vector $\mathbf{a} \in \mathbf{R}^{N-n_G+1}$ and scalar $b \in \mathbf{R}$ are identical to zero in $\mathbf{J}_C[\mathbf{a}^T b]^T = \mathbf{0}$. Thus, from (24), it follows that

$$\begin{aligned} D_\chi\mathcal{F}|_0\mathbf{a} &= -\left.\frac{\partial\mathcal{F}}{\partial\lambda}\right|_0 b \qquad (25)\\ \Rightarrow \rho^T D_\chi\mathcal{F}|_0\mathbf{a} &= -\rho^T \left.\frac{\partial\mathcal{F}}{\partial\lambda}\right|_0 b \qquad \forall\ \rho \in \mathbf{R}^{N-n_G+1} \end{aligned}$$

If $\rho = \varpi$, then equation (25) implies $b = 0$. Therefore, $\mathbf{a}$ must be either cv ($c \neq 0 \in \mathbf{R}$), or $\mathbf{0}$. However, if $\mathbf{a} = cv$, it follows from (24) that $\Delta\chi^T v = \Delta\lambda\,\mathbf{t}_\chi^T v = 0$, which contradicts the assumption of vectors $\mathbf{t}_\chi$ and v being nonperpendicular. Consequently $\mathbf{a} = \mathbf{0}$ and $b = 0$. □

The assumption of vectors $\mathbf{t}_\chi$ and v being nonperpendicular can be justified based on $\mathbf{t}_\chi = cv$ ($c \neq 0 \in \mathbf{R}$) at the bifurcation, as shown ab smoothness of the bifurcation branch of equilibria. The branch of equilibria near a saddle-node bifurcation can be proved smooth using the Lyapunov-Schmidt reduction [31]. Hence, generically, one can expect to fulfill this assumption in a neighborhood of the bifurcation point.

5 Conclusions

This paper presents the conditions needed for detecting saddle-node bifurcations using power flow equations, for a particular differential equation and algebraic constraint model of an ac/dc power system. It is shown that, if the algebraic equations' Jacobian is nonsingular along system trajectories of interest, a necessary condition for having a well posed dynamic system, at a saddle-node bifurcation point the power flow equations meet the same conditions as the reduced differential equations, which are formed by eliminating the algebraic constraints in the model. Also, the paper demonstrates that one only needs to include a more complete static load model in the power flow equations, so that bifurcation points of the proposed system model can be detected using these reduced static equations. The equivalency of saddle-node bifurcation conditions between the differential equations model and the power flow equations, are then used to prove that direct and parameterized continuation methods have nonsigular Jacobians at saddle-node bifurcation points, which makes these techniques powerful computational tools for measuring proximity to points of collapse.

Notice that, although a variety of operational limits are included in the model, it has been assumed throughout this paper that the equations do not change, i.e., the equations and their derivatives are well defined, in the vicinity of a saddle-node bifurcation. Although the condition of an invertible Jacobian associated to the system algebraic constraints is generic, one can always change the models at certain load buses or eliminate Q-limits at some generator buses to guarantee this condition.

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Claudio A. Cañizares (S'87, M'91) was born in Mexico, D.F. in 1960. In April 1984, he received the Electrical Engineer diploma from the Escuela Politécnica Nacional (EPN), Quito-Ecuador, where he is currently a Professor, and the MS (1988) and PhD (1991) degrees in Electrical Engineering from the University of Wisconsin–Madison. Dr. Cañizares' research activities are mostly concentrated in the analysis of voltage stability issues in ac/dc systems.